V. S. Safronov

EVOLUTION OF THE PROTOPLANETARY CLOUD AND FORMATION OF THE EARTH AND THE PLANETS

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AKADEMIYA NAUK SSSR
INSTITUT FIZIKI ZEMLI IMENI O. Yu. SHMIDTA

Academy of Sciences of the USSR
Shmidt Institute of the Physics of the Earth

V.S. Safronov

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(Evolyutsiya doplanetnogo oblaka i obrazovanie Zemli i planet)

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LIST OF SYMBOLS

G — gravitational constant;

H, h — homogeneous width and half-width of cloud (dust layer); in Chapters 1 and 2, H is the magnetic field intensity;

K — angular momentum;

l — impact parameter; in Chapters 2, 3 and 15, mixing length;

m, r, δ — mass, radius and density of growing planet;

P — period of rotation about Sun;

Q — present mass of planet;

R — distance from Sun; in Chapter 15, radius of crater;

T — temperature;

V — velocity of one body relative to another as they approach; in Chapter 12, velocity in nonrotating system linked with Sun;

V_c — Keplerian angular velocity;

v — velocity of body in system moving with angular velocity;

θ — parameter characterizing relative velocities of bodies according to (7.12);

ρ_0 — density in central plane of protoplanetary cloud;

$\rho^* = \frac{3M}{4\pi R^3}$ — density obtained for uniform distribution of mass of central body M within a sphere of radius R;

σ — surface density of matter in zone of planet, i.e., mass in column (of cross section/cm^2) perpendicular to the central plane;

σ' — Boltzmann constant;

τ — time of free flight;

ω — angular velocity.

INTRODUCTION

The origin of the Earth and planets is one of the most involved problems facing science today, and it is one that can be solved only by recourse to many disciplines. An important event in the development of planetary cosmogony over the last twenty years has been the emergence of new sources of data. In the forties Shmidt pointed out that the extensive data pertaining to the Earth sciences should be useful in cosmogony. In the fifties the rich fund of data obtained from physicochemical research on the composition and structure of meteorites and terrestrial rocks was put to use by Urey. In the sixties a new approach appeared — study of the nuclear evolution of protoplanetary matter using data on the distribution and isotopic composition of various elements in meteorites, Earth rocks, and the Earth's atmosphere. Finally, the conquest of the planets and space in the Earth's vicinity has begun to yield new data. Thus planetary cosmogony rests today on a broad spectrum of data collected by related branches of science.

In the present work we will be mainly concerned with the physico-mechanical aspects of the emergence of the planets from the protoplanetary material. Our discussion will be based on the concept of the accumulation of the planets from the solid bodies and particles which, together with the gases, made up the protoplanetary cloud enveloping the Sun in the early phases of its development. The idea that the planets were formed by the condensation of solid bodies and particles — advanced outside the USSR in the 20th century by du Ligondes, Chamberlin and Moulton — was extensively developed in Shmidt's theory and later in the physicochemical studies of meteorites carried out by Urey and others. This idea underlies many geophysical and geochemical studies now being carried out both within and outside the Soviet Union. Foremost among them is the study of the thermal history of the Earth and planets.

In the first part of the book we will study the evolution of the gas-dust protoplanetary cloud and the formation of a cluster of solid bodies. In the second part we will be studying fundamental laws of dynamics in a rotating system of gravitating bodies with inelastic collisions and the process of accumulation of planets by the aggregation of solid bodies and particles. In the third part we develop a method for evaluating the primary temperature of the Earth and discuss the question of primary inhomogeneities in its mantle.

The author is well aware that the idealized schemes considered below do not always depict reality adequately. This is especially true for the early stages of evolution of the protoplanetary cloud which are still very obscure due to the absence of any definite idea on the mechanism of formation of the cloud. However, these idealizations are an essential stage in the development of the theory, for without them it would not be possible to

progress from general qualitative arguments to a concrete quantitative discussion of the probable evolutionary paths. Detailed quantitative study of the different schemes and models can establish which of them are unsuitable and must be discarded, and which are worthwhile and in need of further development. The main requirement to be imposed on this method of research is that far-fetched, abstract schemes totally divorced from reality should not be considered. In choosing the schemes, too, we considered results and data not discussed in this work (research on the early evolution of the stars and Sun; physicochemical, nuclear, and other studies). The picture we draw below of the later stages in the evolution of the protoplanetary cloud and of the formation within it of the planets is the most probable one in our view, given the present level of our knowledge of the solar system. Ideas about the early stages of evolution of the cloud are still vague and subject to change. They are gradually growing more definite as a result of theoretical studies of various cloud models. Valuable information on conditions in the protoplanetary cloud emerges from the study of meteorite structure. Finally, direct observation of the area adjoining other stars is becoming possible. In 1966 Low and Smith carried out infrared observations of the cloud near the star *R* Monoceros, which has features similar to the protoplanetary cloud enveloping our Sun. The chief aim of this work is to give as unified a picture of this evolution as possible in order to be able to subject all aspects to critical discussion and thereby stimulate the development of the theory.

The author thanks Prof. B. Yu. Levin for the many critical remarks offered while the book was being prepared.

Part I

EVOLUTION OF THE PROTOPLANETARY CLOUD AND THE FORMATION OF THE CLUSTER OF SOLID BODIES

Chapter 1

ORIGIN OF THE PROTOPLANETARY CLOUD

1. A few remarks on present-day theories of the origin of the protoplanetary cloud

Since Kant and Laplace, who first put the problem of the origin of the solar system on a scientific footing, many works have been devoted to this problem, and many different cosmogonic hypotheses have been advanced. The need to make sense of the results led to the publication of a series of extensive reviews setting forth the history of cosmogony and analyzing in detail the better known and more interesting hypotheses. We may therefore confine ourselves to remarks concerning contemporary hypotheses still being debated today. Readers interested in the history of cosmogony are referred to the reviews of Jeffreys (1952), Schatzman (1954), Spenser-Jones (1956), Ter Haar and Cameron (1963), Ovenden (1960), Williams and Cremin (1968), and Herczeg (1968). Also, we will not deal with McCrea's hypothesis, which is not as yet advanced enough quantitatively.

Nowadays it is generally accepted that the planets were formed from material rotating about the Sun in the form of an extended gas-dust cloud filling the entire space now occupied by our solar system. The theory of the formation of planets by a gradual accumulation of the solid particles and bodies formed in this cloud has undergone considerable quantitative elaboration in recent years. However, the problem of the origin of the cloud itself has not been solved. Once Laplace's hypothesis of a common origin of Sun and planets in a single nebula had been disproved — together with theories postulating the formation of the planets from matter separated from a formed Sun — it was natural to expect that the idea of capture of the cloud by the Sun would appear in several variants. A hypothesis of gravitational capture was proposed by Shmidt (1944). He based himself initially on an erroneous scheme in which the third body participating in the Sun's capture of material from the interstellar nebula was the center of the Milky Way. Subsequently computations were carried out for capture in the case of three bodies of identical mass, showing that Chazy was mistaken in stating that capture was impossible for a positive energy constant. Evaluating the capture probability, Shmidt (1960) was led to conclude that capture was far more likely at an early stage in the Sun's life, before it had left the parent medium.

Other forms of capture, not purely mechanical, were also proposed; these did not require the proximity of another star. Agekyan (1949, 1950) was able to show with the aid of the accretion mechanism (Bondi and Hoyle, 1944) that the Sun could have captured dust material with the mass and

angular momentum of the planets when it was crossing the edges of the dust cloud. Particles traveling toward the Sun under the pull of its gravity will intersect its trajectory in a sector situated behind the Sun, setting up a high-density region. As a result of inelastic collisions among the particles along this axis their transverse velocity component is damped, leaving only the axial component which is equal to the Sun's velocity relative to the cloud. Particles close to the Sun, for which this velocity is less than the parabolic, are captured by the Sun and begin to revolve around it (instead of landing on the Sun) due to the inhomogeneity of the cloud.

Radzievskii (1950) demonstrated that it was possible for the Sun to capture dust particles of diameter less than 10^{-5} cm due to the reduced role of radiation pressure resulting from shrinking of the particles by evaporation as they approach the Sun. The efficiency of this capture mechanism was not calculated. Alfvén, founder of cosmical electrodynamics and magnetohydrodynamics, suggested capture of the cloud by the Sun with the aid of its magnetic field (1954, 1958). However, a very high magnetic field strength is necessary for this type of capture to be efficient. Lyttleton (1961) surmised the accretion of gas-dust material by the Sun from interstellar clouds of density 10^{-23} g/cm^3 and with the unrealistically low temperature of 3.18°K, rotating with the angular velocity of rotation of our Galaxy, $\sim 10^{-15}$ sec^{-1}. For capture of the required mass with the required angular momentum, the relative velocity of the cloud must be ~ 0.2 km/sec. In the case of a Maxwell distribution of the cloud velocities with an average of $\sim$ 10 km/sec, the fraction of clouds having a velocity ≤ 0.2 km/sec is 10^{-5}. From the Sun's birth to the formation of the planetary system ($\sim$1 billion years), the Sun has encountered interstellar clouds about 10^2 times. From this Lyttleton derives that the probability of capture of protoplanetary material by the Sun is 10^{-3}. However, this estimate is far too high. If the peculiar velocity of the Sun is taken as 20 km/sec, one finds that the fraction of clouds having small velocities relative to the Sun is not 10^{-5} but smaller (10^{-7}). Also, the impact frequency is proportional to the relative velocity. This gives the correction factor 0.02; instead of 10^{-3} we arrive at 10^{-7}.

Woolfson (1964) combined the capture hypothesis with the separation theory of Jeans: the Sun came close to a star of small mass ($\sim 1/7\ M_\odot$) but enormous radius (15 a.u.), and captured some of the material ejected by the tidal swelling of the star. Calculations by Woolfson indicate that the distance to the star at perihelion must be about three star radii. A star of such considerable size can be regarded as in a state of gravitational contraction. Consequently, here too a low probability is a feature of the theory.

Earlier we mentioned that the capture probability turns out to be much larger if one assumes, in view of present-day ideas of the group formation of stars in large clusters of interstellar clouds (Ambartsumyan, 1947; Lebedinskii, 1954; Fowler and Hoyle, 1963), that the Sun was not born in isolation, and if one considers capture during the period of solar genesis when the Sun was still close to other developing stars and nebulae (Shmidt, 1957). It is not possible to state more definitely whether such capture is actually possible before conditions near the growing Sun are investigated.

However, capture theories encounter another difficulty: they fail to explain why the Sun's rotation and the revolution of the planets are in the same sense. One might suppose that part of the captured material landed on the Sun and gave it a spin in the sense of revolution of the cloud. But this

would not correspond quantitatively to the observed distribution of mass and angular momentum in the solar system. The orbital angular momentum of all the planets in the terrestrial group is $2^1/_2$ orders smaller than the angular momentum of Jupiter, while the angular momentum of Mercury is $1^1/_2$ orders smaller than that of the Earth. If the captured material had had such a monotonic variation of angular momentum with distance from the Sun, the angular momentum of that part of the cloud nearer to the Sun than the Mercury zone would have been no greater than the angular momentum of Mercury itself. Therefore, if all this material were somehow to land on the Sun, the angular momentum it would impart would not exceed 0.003% of the angular momentum of all the planets, i.e., it would have been two orders of magnitude less than the rotational momentum of the Sun.

Shmidt (1950) and Radzievskii (1949) conjectured that the Sun could have acquired a rotation in the same sense as the cloud's revolution owing to solid particles landing on the Sun from the cloud due to the Poynting-Robertson effect. Accordingly, we calculated the maximum amount of material that could land on the Sun due to this effect (1955) assuming that the Sun was fully formed and that its mass and luminosity were close to what they are now. The quantity of material landing on the Sun under these conditions due to the Foynting-Robertson effect, as found by us, could have imparted an angular momentum equal to only 0.002 of the present rotational momentum of the Sun. Thus capture theories fail to explain the present rotation of the Sun.

Most astronomers persisted in adhering to classical Laplacian ideas of a common genesis of Sun and cloud; for a long time, however, no concrete common-genesis theory capable of explaining the fundamental laws of the solar system was advanced. The first serious attempt was made after the death of Shmidt. Hoyle (1960) advanced a hypothesis envisaging the common genesis of the Sun and cloud from a single rotating nebula. Using the known relations between the mass and radius of a contracting, rotationally unstable protostar losing material from its equator (Schatzman, 1949; Safronov, 1951)

$$d(\omega M k^2 R^2) = \omega R^2 dM, \quad \omega^2 R^3 \approx GM, \quad M = M_0 (R/R_0)^{2/k^3 - 3} \tag{1}$$

(where kR is the radius of gyration), Hoyle found that there is no purely hydrodynamic mechanism capable of explaining the slow rotation of the Sun. He then suggested that magnetic forces were the leading factor in the transmission of rotational momentum from Sun to cloud. The solar nebula, with initial dimensions of the order of the distance to the nearest stars and an initial angular velocity of the order of that of the Milky Way ($\sim 10^{-15}$ sec^{-1}), was originally tied to interstellar clouds by the galactic magnetic field. At the first stage of slow contraction, in Hoyle's view, it transferred to these clouds the greater part of its rotational momentum. Then, when the material became capable of moving freely across the lines of force, the tie was broken and free gravitational contraction set in, with conservation of angular momentum. When the nebula had contracted to the size of Mercury's orbit it became rotationally unstable. A disk (ring) of mass 0.01 $M_\odot$ separated out in the equatorial region of the nebula. A strong magnetic "bonding" set in immediately between the central condensation (protosun), rotating as a solid body, and the inner edge of the disk (protoplanetary cloud), so that their rotational velocities remained almost

identical and further leakage of material to the disk ceased. The substance in the disk, having acquired angular momentum, began to move away from the Sun and spread throughout the solar system, while the protosun, losing angular momentum, continued to contract. The nonvolatile substances in the disk condensed rapidly into solid particles. The particles were not affected by the magnetic field, but they were carried off by the gas and also spread throughout the solar system. The next process in planet genesis consisted of aggregation of solid particles into large bodies. The energy of rotation of the protosun must have been transformed into magnetic energy. For this to happen, the initial field (~ 1 gauss on separation of the disk*) must have increased in strength to 10^5 gauss, i.e., made 10^5 loops owing to the small difference in rotational momentum remaining between the disk and the protosun. Only a small fraction of this energy was expended in shifting the material of the disk away from the Sun. The greater part must have dissipated within the central condensation. It is possible that a considerable fraction of this energy dissipated in the form of cool (electromagnetic) activity at the surface of the protosun. The latter maintained a level of ionization in the inner region of the disk ($n^+/n > 10^{-7}$) such that there was no perceptible damping of the magnetic field in this region for the entire duration of the process under consideration ($\sim 10^7$ years).

Hoyle's theory was well received and achieved considerable popularity. However, defects gradually emerged. First, according to Hoyle the magnetic field must have transferred angular momentum only to the inner portion of the disk. The transfer of material over large distances from the Sun, in his view, took place by turbulence. But no one has proved that turbulence can exist in a cloud in Kepler rotation and it is doubtful whether it can (see Chapter 2). Such a cloud should be stable with respect to small perturbations, and chaotic motions present from the beginning would apparently be rapidly attenuated (Safronov and Ruskol, 1956, 1957).

Second, Hoyle was very modest in his choice of characteristics for the cloud. The cloud mass 0.01 $M_\odot$ was obtained under the assumption that the compositions of Jupiter and Saturn differ little from that of the Sun. The estimates of other authors lead to a larger mass (see Chapter 12). According to Hoyle, before reaching the stage of gravitational contraction the solar nebula transferred about 99% of its initial angular momentum to the interstellar environment. Cameron (1962) regards this as unlikely. He adduces evidence in favor of a considerably weaker magnetic field in the Milky Way ($3 \cdot 10^{-6}$ gauss) which, in his opinion, precludes significant slowing down of the solar nebula. As a result Cameron arrives at a completely different view of the nebula's evolution (see below).

Third, the process by which Hoyle supposes the solid material to have been transported from the inner edge of the disk (from the distance of Mercury) to the entire solar system — by gases — poses grave difficulties. Hoyle links the motion of gas away from the Sun to tangential (orbital) acceleration of the gas by the magnetic field. The orbital velocity of the gas, in his view, must exceed the circular velocity by a quantity Δv of the order of the radial velocity v_R. The particles are not influenced by the magnetic field; they move with a circular velocity. Therefore the gas must impart to them a tangential acceleration, under the influence of which they move away from the Sun. Hoyle found that in effect the gas shifted away all bodies with a cross section less than 1 m.

* Recently Hoyle and Wicramasinghe (1969) revised the initial field to 10^2 gauss.

However, for a purely tangential acceleration f_φ the deviation Δv in the velocity of the gas from the circular velocity is an infinitesimal of second order with respect to f_φ and v_R (see Safronov, 1960b; or formula (12) below, for $f_R = 0$):

$$\Delta v \approx \frac{3}{2} v_R f_\varphi \frac{R^2}{GM}.$$

Correspondingly, the size of the largest particles carried away by the gas must be much smaller than that found by Hoyle. But the force acting on the gas also has a radial component. According to Hoyle, as the magnetic field twists in the disk the direction of the lines of force tends to become tangential while the direction of the force exerted by the field on the gas tends to become radial. However, the radial acceleration f_R of the gas, by weakening the gravity of the central mass, makes the relative velocity of the gas become smaller than the circular velocity (Whipple, 1964). Therefore the gas does not speed up, but rather slows down the particles, making them draw closer to the Sun. Calculations indicate (Safronov, 1966a) that the particles move away from the Sun only at the initial stage, when they are still small and the field is still untwisted. Their distance from the Sun can increase only by a small fraction of an astronomical unit. Subsequently, due to the twisting of the field and the growth of the particles themselves, the latter begin to draw nearer to the Sun (see Section 3). This result represents a serious stumbling-block for Hoyle's theory: if the protoplanetary cloud separated from the Sun at the distance of Mercury, then the solid particles could not have traveled as far as the positions of the other planets. To resolve this contradiction it would be necessary to revise basic assumptions of the theory.

A different variant of the theory of common genesis was proposed by Cameron (1962). According to Cameron, the magnetic field was important neither at the initial stage of evolution of the protosolar nebula, nor in its collapse. During contraction local angular momentum is conserved and for $R > 100$ a.u. rotational instability sets in, leading to transfer of nearly all the material in the nebula to the disk. After this point, and only after this point, will the magnetic field, which is being twisted in the disk, transmit angular momentum outward, while the inner regions of the disk shift toward the center to form the Sun. Cameron calculated the distribution of density in the disk after collapse for models with an index of polytropy of 1.5 and 3 and mass 4 $M_\odot$ and 2 $M_\odot$, respectively. However, whether such an enormous quantity of material could subsequently leave the solar system remains unclear.

In 1963 Cameron adopted a different model for the disk presupposing that it was formed in the contraction of a homogeneous, slowly turning protostellar cluster. The surface density and angular velocity of rotation of the disk decrease inversely as the distance from the axis of rotation, while the mass is very nearly that of the Sun. Differential rotation intensifies the magnetic field, due to which the greater part of the disk's mass shifts inward to form the Sun, and only a small part, acquiring angular momentum, moves outward. The solid matter which condenses in the latter forms the planets. Later Cameron (1967, preprint) related the transport of material and angular momentum in the disk to turbulence maintained by thermal convection. At temperatures below 2000°K the disk becomes opaque and a

superadiabatic temperature gradient is established at right angles to the plane of the disk. About half of the mass of the disk disperses and therefore its initial mass is again taken as 2 $M_\odot$. According to Cameron, in order for the disk to become opaque and convection to set in, its surface density must be $10^5 - 10^6$ g/cm^2. Convection will give rise to turbulent motions not only along the axis of rotation but also radially. Owing to turbulent viscosity angular momentum is transmitted outward and the disk disperses. The energetic efficiency of this mechanism is still not clear. Further, the cessation of convection which occurs when surface density decreases due to dispersal of the gas to 10^5 g/cm^2 makes it difficult to explain the subsequent evolution of the disk. One should expect gravitational instability to appear inside the disk after its temperature drops to a few hundred degrees, leading to the formation of numerous massive gaseous condensations with a total mass many times greater than that of the planets. It is impossible for such a system of bodies to become our planetary system (see Section 33). Cameron suggests that molecular absorption may have caused the disk to remain opaque and convectively unstable for surface densities of $10^4 - 10^5$ g/cm^2. But if gravitational instability is not to prevail (in the gas) after cooling of the disk, the surface density must be less than 10^4 g/cm^2.

Schatzman (1962) proposed an efficient mechanism of loss of angular momentum by the Sun involving the electromagnetic activity induced on its surface by the interaction between the magnetic field and the convective zone of a rotating star. The material thrown out by the star will be drawn away by its magnetic field and will move with the angular velocity of rotation of the star until a distance is reached where the Coriolis force equals the pressure of the magnetic field: $2\rho V\omega = H^2/4\pi R_c$, where ρ and V are the density and velocity of the ejected matter, H is the magnetic field strength and R_c is the radius of curvature of the lines of force of the magnetic field ($R_c \sim R$ is assumed). The quantity of material ejected is evaluated under the assumption that about 10^{-2} part of the magnetic energy of the active centers is expended in ejection. The loss of matter and angular momentum due to escape of material from the equator of the rotationally unstable star is considered in conjunction with that caused by ejection from the active regions. The loss caused by ejection is initially small but it increases gradually and eventually becomes greater than the outflow. Outflow from the equator ceases while the rotation continues to slow down owing to the continuing ejection of matter from the active regions. The distance of separation, and therefore the angular momentum drawn away by the material, will be slightly smaller if the conditions for the break-up of the bonding between the ejected material and the magnetic field are chosen according to Cowling (1964): $V = v_a = H/\sqrt{4\pi\rho}$, where v_a is the Alfvén velocity.

In 1967 Schatzman gave a more detailed exposition of his nebular theory of the origin of the solar system. He assumes that contraction of the nebula was due not to collapse with free fall velocities, but rather to secular instability. The tremendous energy liberated in contraction could have left the nebula only by convection and turbulence. A uniform rotation was maintained in the nebula owing to the high viscosity. We recall that in Cameron's model opacity and convection appear only after contraction of the nebula and the formation of the disk; during contraction viscosity is low and the angular momentum of the material is conserved. Schatzman describes contraction by equations (1). The outflow of material from the equator

owing to rotational instability proceeded uninterruptedly during contraction of the nebula from the size of the Plutonian orbit to that of Mercury's orbit. For rotationally unstable stars the values of k^2 are several times smaller than for ordinary stars (Auer and Woolf, 1965). For the index of polytropy $n=3$, $k^2=0.038$ and the separated mass is $0.094\ M_\odot$.

It would seem that this value for the mass of the protoplanetary cloud is of the order of the upper limit of admissible values (see Section 32); however, its distribution over distances from the Sun is very different from the distribution of planetary material. If to the latter one adds the light elements required to equal the cosmic (solar) composition, it will be found that the surface density σ of material in the solar system is approximately constant up to Jupiter, dropping off only after Jupiter as R^{-2}. In Schatzman's model $\sigma \propto R^{-2}$ throughout the cloud and the cloud mass within the limits 0.3 — 3 a.u. is equal to the cloud mass within the limits 3 — 30 a.u.

On the other hand, for the isotropic turbulence adopted by Schatzman gases behave as ideal gases with $\gamma = {}^5/_3$. But then the index of polytropy should be $n = 1.5$ rather than 3, and the mass of the protoplanetary cloud separated from the equator of the protosun would be too large ($0.46\ M_\odot$).

Schatzman belives that no less than half of the deuterium present in the Earth and in meteorites was formed inside the protoplanetary cloud when helium nuclei were split by cosmic rays emitted by the active protosun. But if this is so, then within the period when the cloud was being irradiated by cosmic rays ($\sim 2 \cdot 10^6$ years) hydrogen must have escaped from the region of terrestrial planets. Schatzman has shown that this is possible. It is still not clear, however, how hydrogen could have been preserved in the zones of Jupiter and Saturn, where the formation of massive bodies capable of holding hydrogen would require a considerably greater interval of time.

Many aspects of the growth of the Sun and protoplanetary cloud from a single nebula still remain vague or obscure. This is particularly true of the role of the magnetic field in the process. Nonetheless, at this time the idea of a common formation is more promising than that of capture. In view of recent achievements along these lines one may hope that there will be further progress toward a solution of the problem of the origin of the protoplanetary cloud, in close conjunction, of course, with advances in stellar cosmogony. As it happens, the theory of planet formation in the protoplanetary cloud by the accumulation of solid material antedates first attempts to broach the question of the origin of the cloud itself. By establishing the distinction between these two branches of planetary cosmogony Shmidt advanced work on the theory by fifteen years. At the time, indeed, the fundamental laws governing the process of stellar formation were obscure. Data on the Earth's chemical composition (deficit in inert gases, etc.) and on the composition and structure of meteorites pointed to the formation of the planets out of solid material. Planet formation depends directly on the problem of the cloud's origin only at the earliest stage, that of formation of solid bodies from diffuse material. The second stage, the aggregation of solid bodies into planets, displays its own typical laws which to a large extent efface the previous evolution of the cloud. Investigation of these laws has made it possible to draw a picture of planetary formation and relate it to the subsequent state of the planets.

An investigation of the early evolution of the cloud must rest on concrete premises regarding its origin. The cloud model we will consider is closest to that adopted in Schatzman's theory. The initial mass of the cloud is about

0.05 $M_{\odot}$. Solid particles are not carried away by gaseous flows on the periphery of the solar system as assumed in Hoyle's theory. The contraction of the protosun and its active stage lasted about 10^7 years (Fowler and Hoyle, 1963; Ezer and Cameron, 1965). The interval of cloud formation (contraction to dimensions of Mercury's orbit), as well as the period of formation of the protoplanetary bodies within the cloud, was considerably shorter. Bodies of asteroidal size developed within 10^7 years; irradiation of small bodies and particles by the active protosun had a decisive influence on their chemical and nuclear evolution. On the other hand, the rate of the cloud's evolution and the rate of accumulation of the bodies vary with distance from the Sun. When discussing the cloud's evolution below we shall therefore have in mind the sequence of events at a definite distance from the Sun, and not events unfolding simultaneously throughout the cloud.

2. The angular momentum acquired by the Sun due to the Poynting-Robertson effect

We saw in Section 1 that theories involving capture of the protoplanetary cloud by the Sun fail to explain why the Sun rotates in the same sense as the planets around it. It has been suggested that the Sun's rotation is due to material landing on it from the cloud owing to the Poynting-Robertson effect. We now proceed to evaluate this effect, demonstrating its inadequacy (Safronov, 1955).

Particles moving around the Sun are slowed down by its radiation; their orbits shrink and they slowly draw closer to the Sun. This phenomenon, known as the Poynting-Robertson effect, is quite simple to describe if one considers an isolated particle being acted upon only by the forces of the Sun's gravity and radiation. In the protoplanetary cloud the particles lay in a gas and their motion depended essentially on the motion and density of the gas. If the only force acting on the gas was that of the Sun's gravity, it must have moved with a Keplerian angular momentum greater than the circular velocity of the particles which were subjected to the pressure of the Sun's radiation. The gas accelerated the particles in the direction of their motion and they drew farther from the Sun. But given the twisting of the Sun's magnetic field, the circular velocity of the gas must have been less and the particles must have drawn nearer to the Sun (see Section 3). After the gas dispersed, the particles also moved toward the Sun. We do not understand conditions in the cloud well enough to make definite statements about the particles' motion. Once they reached a distance of 0.03 a.u. from the Sun, the solid particles evaporated. Having absorbed light quanta and experienced particle collisions, the molecules lost part of their orbital angular momentum and came close to the Sun. It seems the corpuscular impacts were more efficient. But in this process some of the molecules acquired high velocities and left the area of the Sun. It is still not clear what fraction of the molecules landed on the Sun's surface. However, the fact that the actual efficiency of the mechanism under consideration is indeterminate does not prevent one from evaluating the upper limit of the angular momentum which the Sun could have acquired due to the Poynting-Robertson effect. The flow of matter to the Sun was determined only by the quantity of solar radiation trapped in the cloud. Consider two simple schemes: the motion of isolated

particles reemitting solar radiation, and the combined motion of the particles and gas which trap incoming solar corpuscles.

A particle of mass m moving around the Sun along a circular orbit of radius R, upon isotropic reemission of the incident light of mass dm_r, will lose the angular momentum $\sqrt{GMR}dm_r$, where M is the mass of the Sun. The angular momentum imparted to the particle by radiation is negligible. With reduction of its angular momentum the radius of the particle's orbit will decrease:

$$md\sqrt{GMR} = -\sqrt{GMR}dm_r. \tag{2}$$

Integrating (2) we obtain the reduction of the orbital radius of the particle m due to the radiation of mass m_r incident upon it:

$$R = R_0 e^{-\frac{2m_r}{m}}. \tag{3}$$

Corpuscles trapped in the cloud produce a slightly different effect. Assuming that the particles move together with the gas, let us consider the motion of a certain elementary volume of the cloud around the Sun. As in the previous instance, the distance of this volume from the Sun is determined by the law of conservation of angular momentum. For a volume with initial mass m and initial circular orbit of radius R_0, disregarding the angular momentum of the corpuscles, we obtain

$$(m + m_e)\sqrt{GMR} = m\sqrt{GMR_0} \tag{4}$$

and

$$R = \left(\frac{m}{m + m_e}\right)^2 R_0, \tag{5}$$

where m_e is the mass of corpuscular emission trapped in this volume.

Table 1 gives the values of m_r/m and m_e/m for R_0 equal to the distances of various planets from the Sun and R equal to the Sun's radius.

TABLE 1

	Mercury	Venus	Earth	Mars	Jupiter
m_r/m	2.2	2.5	2.7	2.9	3.5
m_e/m	8	11.5	14	17	32

Of the entire solar emission of mass ΔM, the cloud absorbs the mass $\Omega\Delta M/4\pi$, where Ω is the solid angle subtended at the Sun by the opaque portion of the cloud. This causes the following mass of material to land on the Sun:

$$\Delta m = \alpha_1 \frac{\Omega}{4\pi} \Delta M = \alpha_2 \Delta M. \tag{6}$$

In the case of photon emission α_1 is equal to the ratio m/m_r, the inverse of which is given in Table 1. For the inner edge of the protoplanetary cloud one can assume $m/m_r \approx 1/2$. In the case of corpuscular emission, the Sun receives not only the cloud material m but also the trapped corpuscular emission of mass m_c, which is one order greater. For α_1 we obtain the value $\alpha_1 = \frac{m + m_c}{m_c} \approx 1.1$. Thus for photon emission $\alpha_1 \approx 1/2$ and for corpuscular emission $\alpha_1 \approx 1$.

Since not all corpuscles landing in the cloud are trapped inside it, it seems that in the second case α_1 is less than unity and not very different from its value for photon emission. For a relatively flat cloud having thickness H at distance R from the Sun, $\Omega/4\pi \approx H/2R$. For the usually accepted value $H \approx 1/25\ R$ and $\alpha_1 = 1/2$, we have $\alpha_2 = 0.01$. The mass Δm reaching the Sun from the cloud in 10^8 and 10^9 years due to solar photon emission — at the present rate (total emission $= 4 \cdot 10^{12}$ g/sec $= 1.2 \cdot 10^{29}$ g in one billion years) — is given in Table 2 for several values of α_2. The corresponding angular momentum $\Delta K = \Delta m \sqrt{GMR_\odot}$ imparted by this material is given in the second row, in terms of the present angular momentum of the Sun $K_\odot$. For $\alpha_2 = 0.01$, about 0.1 of the mass of the inner planets lands on the Sun from the cloud in 10^9 years. The angular momentum imparted to the Sun by this material amounts to only 0.002 of the present solar angular momentum. For the Sun to have acquired its present rotation, the added mass should have amounted to 10^2 Earth masses. Therefore, the Poynting-Robertson effect could have caused the present rotation of the Sun only for a solar radiation L two to three orders of magnitude greater than the present value (see last row of Table 2).

TABLE 2

$\alpha_2 = \alpha_1 \frac{\Omega}{4\pi}$	0.001		0.01		0.1	
Δt, years . .	10^8	10^9	10^8	10^9	10^8	10^9
Δm, g . . .	10^{25}	10^{26}	10^{26}	10^{27}	10^{27}	10^{28}
$\Delta K/K_\odot$ . .	$2 \cdot 10^{-5}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-4}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-2}$
$L/L_\odot$ . . .	$6 \cdot 10^4$	$8 \cdot 10^3$	$6 \cdot 10^3$	$6 \cdot 10^2$	$6 \cdot 10^2$	$6 \cdot 10^1$

The Sun's luminosity may have been two orders of magnitude greater than today in the closing stage of gravitational contraction (Hayashi, 1962). But the duration of this stage did not exceed 10^7 years, and the total amount of radiation lost by the Sun in this period did not exceed the radiation lost by the Sun in 10^9 years. Consequently, even if the Sun's capture of the cloud began before the onset of the high-luminosity phase, when the Sun was still newly formed, the Poynting-Robertson effect could not have imparted the required rotational angular momentum.

Vinogradova (1961) has shown that the rotation of the particles makes it necessary to allow for the anisotropy reemission of solar radiation by them. A particle rotating in the same direction as the Sun will emit less in the direction of motion along its orbit than in the opposite direction. In the process it will acquire positive angular momentum and move away from the Sun. A particle rotating in the opposite direction will draw closer to the Sun. The effect is maximum for a certain velocity of rotation determined by the

dimensions and physical properties of the particle. In this case it is stronger by three orders of magnitude than the Poynting-Robertson effect. Assuming maximum efficiency, this mechanism could have caused a substantial redistribution of angular momentum in the solar system, as suggested by Vinogradova, and could also have given the Sun its present rotation. But in reality its efficiency must have been considerably below the maximum, first because the rotational velocities of the particles vary (only in very rare cases do they approach values for which the effect is maximum), and second because particle collisions caused the magnitude and direction of the rotational velocities to alter sharply, direct rotation being replaced by inverse rotation and vice versa. Thus the motion of particles along R was nondirectional — of the nature of random flight, or diffusion — and the distance of a particle from its initial position increased not proportional to the time t but to $\sqrt{t}$.

3. Motion of solid particles in a gas driven by the magnetic field

We assume that in the absence of a magnetic field in the protoplanetary cloud every volume element of the cloud travels around the Sun along a circular orbit with a Kepler velocity. From the entire magneto-hydrodynamic problem, we will consider only the one-sided action of the magnetic field on the motion of the gas. Suppose that this effect is expressed in the presence of the radial and tangential accelerations f_R and f_φ of the gas. Disregarding the gas pressure gradient and viscosity, we can write the equations of axisymmetric motion (Landau and Lifshits, 1953) as

$$\frac{\partial v_R}{\partial t} + v_R \frac{\partial v_R}{\partial R} - \frac{v_\varphi^2}{R} = -\frac{GM}{R^2} + f_R, \tag{7}$$

$$\frac{\partial v_\varphi}{\partial t} + v_R \frac{\partial v_\varphi}{\partial R} + \frac{v_R v_\varphi}{R} = f_\varphi. \tag{8}$$

For small f_R and f_φ the motion can be treated as stationary and nearly circular. Then

$$\frac{\partial v_R}{\partial t} = \frac{\partial v_\varphi}{\partial t} = 0; \quad v_\varphi = V_e + \Delta v; \quad V_e = \sqrt{\frac{GM}{R}}. \tag{9}$$

From (9) we obtain

$$v_R = \frac{f_\varphi}{\frac{\partial v_\varphi}{\partial R} + \frac{v_\varphi}{R}} = \frac{2f_\varphi}{\omega}\left[\frac{1}{1 + \frac{2\Delta v}{v_e} + \frac{2}{\omega}\frac{\partial \Delta v}{\partial R}}\right]. \tag{10}$$

Equations (7) and (8) yield

$$v_\varphi^2 = V_e^2 + 2V_e\Delta v + (\Delta v)^2 = v_r R \frac{\partial v_R}{\partial R} + \frac{GM}{R} - Rf_R \tag{11}$$

and

$$\Delta v = -\frac{f_R}{2\omega} + v_R \frac{R\frac{\partial v_R}{\partial R}}{2V_c} - \frac{(\Delta v)^2}{2V_c}. \tag{12}$$

Retaining only first-order terms, we have

$$v_R \approx \frac{2f_\varphi}{\omega}, \qquad \Delta v \approx -\frac{f_R}{2\omega}. \tag{13}$$

Thus Δv is practically always negative. Under the influence of the magnetic field the gas moves around the Sun with a velocity less than the circular Kepler velocity. Only for $f_R \ll f_\varphi$ is $\Delta v > 0$, having the form (7) and being a second-order infinitesimal.

The motion of dust particles can be determined from the same arguments, except that f_φ and f_R must be replaced in (13) by the perturbing accelerations g_φ and g_R acting on the particle from the gas:

$$g_\varphi = C\omega(\Delta v - \Delta v_p), \qquad g_R = C\omega(v_R - v_{pR}), \tag{14}$$

where $C = 2\sigma_g/\pi r\delta$; r and δ are the particle radius and density, σ_g is the surface density of the gas, and v_{pR} and Δv_p are the components of the particle velocity. Therefore

$$v_{pR} \approx \frac{2g_\varphi}{\omega} = 2C(\Delta v - \Delta v_p); \qquad \Delta v_p \approx -\frac{g_R}{2\omega} = -\frac{C}{2}(v_R - v_{pR})$$

and finally

$$v_{pR} = \frac{C^2}{(1+C^2)\omega}\left(2f_\varphi - \frac{f_R}{C}\right); \qquad \Delta v_p = \frac{C^2}{(1+C^2)\omega}\left(\frac{f_\varphi}{C} + \frac{f_R}{2}\right). \tag{15}$$

Small particles for which $C^2 \gg 1$ and $2Cf_\varphi \gg f_R$ practically move together with the gas. Particles for which $2Cf_\varphi < f_R$ travel to the Sun. The corresponding condition for the particle size has the form

$$r > r_0 = \frac{4\sigma_g}{\pi\delta}\operatorname{ctg}\theta, \tag{16}$$

where θ is the angle between the radius vector and the line of force. Directly after separation of the disk the magnetic field is still untwisted; $\operatorname{ctg}\theta \sim 1$ and for $\sigma_g \sim 10^3$ the value of r_0 is of the order of several meters. In practice all the particles are driven off by the gas. However, within a few tens of years twisting of the field causes r_0 to shrink to a few centimeters. Within this time many particles grow to the radius $r > r_0$ and begin to draw closer to the Sun, having traveled only a small fraction of an astronomical unit away from it. As the field twists its intensity increases. In the process, $\theta \to \pi/2$ and $\operatorname{ctg}\theta \sim n^{-1}$, where n is the number of loops. When, according to Hoyle, n reaches 10^5 the critical radius r_0 will be less than 10^{-2} cm; practically all particles will begin to move under the influence of the gas toward the Sun, and not away from it, as supposed by Hoyle.

Thus, it seems that the transfer of solid particles by the gas, assumed in Hoyle's theory to have taken place from the Mercury zone to the orbits of the other planets, is not possible.

Chapter 2

TURBULENCE IN THE PROTOPLANETARY CLOUD

4. Condition of convective instability in rotating systems

The problem of turbulence, or more precisely of the chaotic macroscopic motions which may have appeared in the protoplanetary cloud during its formation, is vital for understanding the cloud's evolution. The character of subsequent processes in the cloud must have depended on whether these primordial motions were damped or whether some kind of stationary turbulence was established in the cloud material. The persistence of turbulence would have prevented the separation of dust and gaseous components and the formation of dust condensations. Planetary nuclei could then have been formed only by the direct growth of solid particles due to agglomeration in collisions.

The idea of turbulence was introduced to cosmogony by von Weizsäcker, by way of a return to Descartes' classical eddies (1944). Von Weizsäcker pointed out that the Reynolds number for a cosmic diffuse medium is very large, far above the critical value. Since then turbulence has been regarded as one of the most widespread states of cosmic matter. Von Weizsäcker also conjectured that turbulence was important in the formation of the heavenly bodies and their systems as well. In his view the planets, stars, Milky Way and other structures were formed from turbulent eddies of eration in collisions.

For the protoplanetary cloud the Reynolds number $\mathrm{Re} = \frac{\rho l v}{\eta}$ was greater than 10^{10}. To explain the law of planetary distances, von Weizsäcker assumed that the turbulent motions within the cloud constituted a regular system of eddies whose sizes were proportional to distance from the Sun. Without going into a detailed analysis of all the tenets of von Weizsäcker's theory, we consider only the fundamental idea of the prolonged persistence ($\sim 2 \cdot 10^8$ years) of turbulence in the protoplanetary cloud.

In rotating systems the Reynolds number is not the paramount criterion of stability of motion. If one wishes to understand the motion of matter in the protoplanetary cloud, which was a fairly flat system, one can turn to results of studies of the motion of a fluid between two rotating cylinders (Couette flow). According to Rayleigh's well-known criterion for incompressible inviscid fluids (1916), the necessary and sufficient condition for stability of a purely rotational motion with angular velocity $\omega(R)$ is

$$\frac{d}{dR}(\omega R^2)^2 > 0 \qquad (1)$$

throughout the fluid under consideration. Instability arises if this condition is violated anywhere. Rayleigh's criterion was confirmed in theoretical and experimental studies by Taylor (1923). It was found that a liquid's viscosity increases its stability. Chandrasekhar pointed out that while the fluid is necessarily stable if Rayleigh's criterion (1) is met, it is not necessarily unstable if Rayleigh's criterion is not met (1958). The same author carried out a theoretical analysis of the problem of stability for the more general case where the distance between cylinders is not small compared with the radius. The calculations for $R_2/R_1 = 2$ were confirmed by experiments of Donnelly and Fultz (1958, 1960).

According to Rayleigh's criterion (1), the protoplanetary cloud should be stable. For circular Kepler motion the angular momentum is proportional to $\sqrt{R}$, i.e., increases with R, and the stability condition (1) is met. The gas pressure within the cloud is low, and the motion should be almost Keplerian. Disregarding the gas pressure gradient dp/dR, condition (1) applied to the flat protoplanetary cloud reduces to the condition of stability for circular orbits known from stellar dynamics (see Chandrasekhar, 1942):

$$\frac{\partial}{\partial R}\left(R^3 \frac{\partial \Phi}{\partial R}\right) > 0, \tag{2}$$

where Φ is the potential energy at the distance R from the axis of rotation (and axis of symmetry) of the system. The mass of the cloud is small compared with the solar mass, and gravitation is determined predominantly by the central body, i.e., $d\Phi/dR = GM/R^2$. Thus condition (2) is met and therefore circular orbits in the protoplanetary cloud are stable.

However, conditions (1) and (2) fail to allow for the possibility of convection appearing in the cloud. Von Weizsäcker attempts to substantiate the persistence of turbulence in revolving cosmic gaseous masses, including the protoplanetary cloud, by means of the condition for the appearance of convection (1948). But he disregards the rotation and does not take the stability condition (1) into account. Concurrent analysis of these conditions led to the following results (Safronov and Ruskol, 1956, 1957).

Suppose a liquid is in a purely rotational laminar motion $\omega(R)$, where R is the distance from the axis of rotation. The sum of the forces (gravitational, centrifugal and pressure) acting radially upon any element of the liquid is zero:

$$f = -\frac{GM}{R^2} + \omega^2 R - \frac{1}{\rho}\frac{dp}{dR} = 0. \tag{3}$$

Now if we apply to this element the small perturbation δR, conserving its angular momentum ωR^2, it will be acted upon by the following force per unit mass:

$$\delta f = \frac{2GM}{R^3}\delta R - 3\omega^2 \delta R - \left(\frac{1}{\rho_{el}}\frac{dp}{dR}\right)_{R+\delta R} + \frac{1}{\rho}\frac{dp}{dR}. \tag{4}$$

The subscript "el" indicates that the given characteristic refers to the element under consideration. The motion is stable if the direction of the force δf is opposite to the direction of displacement, i.e., $\delta f < 0$ for $\delta R > 0$. Since (3) is satisfied for all R, its derivative along R is zero:

$$f'\delta R = \frac{2GM}{R^3}\delta R + (\omega^2 R)'\delta R - \left(\frac{1}{\rho}\frac{dp}{dR}\right)_{R+\delta R} + \frac{1}{\rho}\frac{dp}{dR} = 0. \tag{5}$$

Subtracting (5) from (4), we obtain

$$\delta f = -[3\omega^2 + (\omega^2 R)']\delta R - \frac{dp}{dR}\left(\frac{1}{\rho_{el}} - \frac{1}{\rho}\right)_{R+\delta R}. \tag{6}$$

At the distance R, i.e., for an unperturbed element, $\rho_{el} = \rho$. Therefore

$$\left(\frac{1}{\rho_{el}} - \frac{1}{\rho}\right)_{R+\delta R} = -\frac{1}{\rho^2}\left[\left(\frac{d\rho}{dR}\right)_{el} - \frac{d\rho}{dR}\right]\delta R. \tag{7}$$

Furthermore,

$$3\omega^2 + (\omega^2 R)' = 2\frac{\omega}{R}(\omega R^2)'. \tag{8}$$

Therefore the condition for stability of rotational motions with respect to convection $\delta f/\delta R < 0$ has the form

$$\frac{d}{dR}(\omega R^2)^2 > \frac{R^3}{\rho^2}\frac{dp}{dR}\left[\left(\frac{d\rho}{dR}\right)_{el} - \frac{d\rho}{dR}\right]. \tag{9}$$

When the right-hand side is zero this condition reduces to Rayleigh's criterion (1), and when the left-hand side is zero to the ordinary condition for the onset of convection in a nonrotating medium.

If the element of volume is moving adiabatically (the adiabatic index $\gamma = c_p/c_v$), then

$$\left(\frac{d\rho}{dR}\right)_{ad} = \frac{1}{\gamma}\frac{\rho}{p}\frac{dp}{dR}, \qquad \frac{d\rho}{dR} = \frac{\rho}{p}\frac{dp}{dR} - \frac{\rho}{T}\frac{dT}{dR} \tag{10}$$

and

$$\left(\frac{d\rho}{dR}\right)_{ad} - \frac{d\rho}{dR} = \frac{\rho}{\gamma T}\left[\frac{dT}{dR} - (\gamma - 1)\frac{T}{\rho}\frac{d\rho}{dR}\right]. \tag{11}$$

The condition for the onset of convection has the form*

$$\frac{dT}{dR} \lesseqgtr (\gamma - 1)\frac{T}{\rho}\frac{d\rho}{dR} + \frac{\gamma T \rho}{R^3 dp/dR}\frac{d}{dR}[\omega R^2]^2 \quad \text{for} \quad \frac{dp}{dR} \gtrless 0. \tag{12}$$

The right-hand side represents the adiabatic gradient in a revolving medium.

For small displacements δR the smoothly varying functions ρ, p and T can be approximated by power functions

$$\rho \sim R^{-a_1}; \qquad p \sim R^{-a_2}; \qquad T \sim R^{-a_3}, \tag{13}$$

* This condition could have been expressed in the more usual form $\frac{dT}{dR} < -\frac{g}{c_p} - \frac{T}{gR^3}\frac{d}{dR}(\omega R^2)^2$, where $g = \frac{GM}{R^2} - \omega^2 R$, but it is less convenient for quantitative calculations.

where $a_3 = a_2 - a_1$. Then, taking ωH^2 from (3), we can reduce the convection condition (12) to the simple expression

$$a_3 > (\gamma - 1) a_1 + \frac{\gamma}{a_2} \frac{GM\mu}{R\mathfrak{R}T} + \gamma(2 - a_3)$$

or

$$2a_3 > \frac{\xi}{a_2} + (1 - \gamma^{-1}) a_2 + 2, \tag{14}$$

where $\xi = \frac{GM\mu}{R\mathfrak{R}T}$; $\mathfrak{R}$ is the gas constant and μ the molecular weight.

The minimum $\xi = \xi_0$ in the right-hand side of (14) will be obtained if we assume that the protoplanetary cloud is transparent and that its temperature is that of a black sphere at the corresponding distance from the Sun. Then for the present solar luminosity

$$T_0 \approx \frac{300^\circ}{\sqrt{R_{\text{a.u.}}}}, \tag{15}$$

where $R_{\text{a.u.}}$ is the distance in astronomical units and

$$\xi_0 \approx \frac{3 \cdot 6 \cdot 10^2 \mu}{\sqrt{R_{\text{a.u.}}}} \approx \frac{10^3}{\sqrt{R_{\text{a.u.}}}}. \tag{16}$$

Clearly, then, the condition for convection (14) could not have been met in any significant part of the protoplanetary cloud. The coefficient a_2, which is related to the pressure gradient, cannot be large. Therefore from (14) convection would require a very high temperature gradient ($a_3 \sim \xi$). This is excluded in a transparent cloud since from (15) $a_3 = 1/2$. A high temperature gradient could exist along the innermost edge of an opaque cloud provided the cloud boundary was sharp. The width of the resulting zone would obviously be very small ($\sim R/a_3 < 10^{-2}$ a.u.). Convection in such a narrow zone could not have had a perceptible effect on the cloud's dynamics. However it might have increased the width of the inner edge, causing screening of solar radiation by the solid particles, a drop in the cloud temperature, and therefore alteration of the chemical composition of the particles (see Chapter 4). However, a sharp inner boundary is ruled out by the Poynting-Robertson effect (see Chapter 1). According to Fesenkov (1947), in transparent solar space the density of a stationary stream of particles of uniform size moving under the influence of this effect toward the Sun varies as R^{-1}. But along the edge of an opaque cloud the density of the dust component must have decreased toward the Sun owing to two factors: the radiation density increased faster than R^{-2} due to the falling off of absorption, and the particles shrank by evaporation. Therefore the temperature gradient necessary for convection could not have become established in this band either.

Thus the revolving protoplanetary cloud was stable with respect to small perturbations, convection could not have set in inside it, and von Weizsäcker's conjecture regarding turbulence due to convection has not been confirmed.

5. Other possible causes leading to disruption of stability

The formation of the solar cloud was not a smooth process, and primordially the cloud may have contained random macroscopic motions of large scale. The cloud's stability with respect to small perturbations, demonstrated above, does not necessarily imply that these random motions were damped quickly.* Therefore the question of the possible persistence of "turbulence" in the cloud needs to be discussed further.

If the energy of the primordial random (i.e., with respect to the circular Kepler velocity) motions was not dissipated, these motions would persist for an indefinite time. This is the property with which von Weizsäcker endowed his ordered system of eddies. All the particles in an eddy move along Kepler ellipses with the same period and without loss of energy; the center of the eddy moves along a circular orbit. It would have been more natural to suppose that the centers of the eddies move along elliptical orbits. But then there would inevitably have been dissipation of the energy of relative motion — rounding out of the orbits during motion in a resisting medium and consequent damping of turbulence.

We could assume, in common with von Weizsäcker (1944), that the energy source which maintains turbulence inside the cloud is the gravitational energy of the cloud in the Sun's gravitational field, liberated as the inner regions of the cloud approach the Sun. Von Weizsäcker (1948) and Lüst (1952) describe the evolution of the revolving turbulent cloud by means of the ordinary equations of hydrodynamics, merely replacing molecular by turbulent viscosity $\eta=\rho l v$. For the large-scale turbulence they assume the value of η is large and the equations predict a highly efficient transport of material (from outer regions of the cloud outward and from inner regions sunward).** This purely formal application of hydrodynamics to turbulent motion, however, is not justifiable. The mixing length l is comparable with the dimensions of the system ($l \approx 0.6\,R$) and the velocity distribution is not Maxwellian. The authors take shearing stresses dependent, as usual, on the angular velocity gradient:

$$\sigma'_{R\varphi} = \eta R \frac{\partial \omega}{\partial R}. \tag{17}$$

In Prandtl's semiempirical theory of turbulence the shearing stresses are assumed to depend on the gradient of angular momentum:

$$\sigma'_{R\varphi} = \frac{\eta}{R} \frac{\partial}{\partial R} (\omega R^2). \tag{18}$$

The above relation is also used by von Karman (1953). This distinction is very important for the evolution of the protoplanetary cloud since angular velocity inside it decreases away from the Sun while angular momentum increases, i.e., transport takes place in opposite directions depending on the point of view. Taylor's experiments (1923) tend to favor the Prandtl-von Karman view, although the Prandtl theory is excessively simplified and

* Here too an analogy can be drawn with the motion of fluids between two revolving cylinders. Experiments show that there exists a region of fairly large Re in which stationary motion is metastable: it is stable with respect to small perturbations but unstable with respect to large ones (Landau and Lifshits, 1953).

** Analysis of the energies involved (Ter Haar, 1950) has shown that the decay time of such a cloud ($10^2 - 10^3$ years) is five orders of magnitude less than the time required (according to von Weizsäcker) for the planets' growth.

the real picture of turbulent motion is far more complex. Vasyutinskii (1946) has put forward a more general expression for the shearing stresses in a revolving medium:

$$\sigma'_{R\varphi} = \rho \frac{K_R^R}{R} \frac{d}{dR}(\omega R^2) - 2\rho K_\varphi^\varphi \omega, \tag{19}$$

where K_R^R and K_φ^φ characterize the mean transfer vl in the radial and transversal directions. For isotropic transfer ($K_\varphi^\varphi = K_R^R$) it reduces to the ordinary hydrodynamic expression (17) and for purely radial transfer ($K_\varphi^\varphi = 0$) to the expression (18) of Prandtl. For the protoplanetary cloud ($\omega \sim R^{-3/2}$) Vasyutinskii's relations lead to damping of turbulence when $K_\varphi^\varphi < \frac{1}{4} K_R^R$; however, they do not make it possible to evaluate either the ratio $K_\varphi^\varphi / K_R^R$ or the scale of the turbulence. Moreover it is not clear to what extent these generalizations are physically justified. However, Vasyutinskii's conjecture that expressions (17) and (18) for $\sigma'_{R\varphi}$ correspond to two extreme cases and that for real turbulent motion $\sigma'_{R\varphi}$ should be describable by an intermediate relation appears to be reasonable.

Using $\sigma'_{R\varphi}$ it is possible to estimate how much energy of ordered rotational motion (replenished in turn by potential energy in the gravitational field of the central mass) is converted by viscosity into energy of random motion. Expression (17) gives us the amount of energy converted into heat per cm^3 per second for laminar rotational motion of a fluid (Lamb, 1932):

$$E = \eta R^2 \left(\frac{d\omega}{dR}\right)^2, \tag{20}$$

where $\eta = \frac{1}{3}\rho v \lambda$ and the mean free path $\lambda \ll R$. Similarly, expression (18) deriving from Prandtl's theory gives the amount of energy converted into turbulent motion:

$$E = \frac{\eta}{R^2}\left[\frac{d(\omega R^2)}{dR}\right]^2, \tag{21}$$

where $\eta = \frac{1}{3}\rho v l$ is the turbulent viscosity, l the mixing length, and v the turbulent velocity. For Kepler rotation ($\omega \sim R^{-3/2}$) the above expression differs from (20) only by the factor $1/9$. Assuming that the correct value of E for turbulent motion lies between (20) and (21), we can take expression (20) as our basis and introduce the factor $1/9 < \beta' < 1$ in the right-hand side. In Chapter 7 this method is used to estimate the velocity dispersion in a system of gravitating bodies with Kepler rotation, since for large free paths the nature of the transfer should be the same as in turbulent motion. The value $\beta' \approx 0.2$ was obtained.

Thus the turbulent energy acquired per unit mass per second can be taken as $\varepsilon' = \beta' E/\rho$. The turbulent energy which dissipates (converting into thermal energy) is $\varepsilon'' \approx \frac{v^2}{2\tau} \approx \frac{v^3}{2l}$ per gram per second in well-developed turbulence. By comparing ε' with ε'' it is possible to determine whether the turbulence is being enhanced or, on the contrary, damped:

$$\varepsilon = \varepsilon' - \varepsilon'' = \frac{v^2}{2\tau}\left(\frac{3\beta'}{2}\frac{l^2\omega^2}{v^2} - 1\right) = \frac{v^2}{2\tau}\left(\frac{3}{2}\beta'\tau^2\frac{4\pi^2}{P^2} - 1\right), \tag{22}$$

where τ is the mean free time of the eddies and P is the formation period.

Expression (22) holds only for $\tau \ll P$. In this case $\epsilon < 0$. Consequently such small-scale turbulence must die down. For large τ, l^2 must be replaced by $^1/_2\overline{\Delta R^2}$, where ΔR is the change in the distance R of the eddy during time τ. Let us assume that one-third of the eddies has only a radial component of relative velocity $v = v_R$, one-third has $v = v_\varphi$, and one third has $v = v_z$. For eddies with $v = v_z$, $\Delta R = 0$. For eddies with $v = v_R$ and $v = v_\varphi$ we will take, in accordance with (7.24)* and (7.25),

$$\overline{\Delta R_R^2} = \frac{1}{2} R^2 e^2, \quad \overline{\Delta R_\varphi^2} = \frac{3}{2} R^2 e^2. \tag{23}$$

Since the first term of (22) already contains a factor $^1/_3$ in the expression for η, the sum $\overline{\Delta R_R^2} + \overline{\Delta R_\varphi^2}$ should be used for $\overline{\Delta R^2}$. We take $v^2 = \frac{9}{8} e^2 V_c^2$ in accordance with (7.28). Then for large τ

$$\epsilon = \frac{v^2}{2\tau}\left[\frac{3}{2}\beta' \frac{8}{9} \frac{e^2 R^2 \omega^2}{e^2 V_c^2} - 1\right] = \frac{v^2}{2\tau}\left(\frac{4}{3}\beta' - 1\right). \tag{24}$$

The value $\beta' = 0.2$ obtained in Chapter 7 yields $\epsilon < 0$. Thus the turbulence must subside. For turbulence to persist it is necessary that $\beta' > \frac{3}{4}$, which is apparently unrealistic.

In the above we have taken $\varepsilon'' = v^3/2l$ under the assumption that the energy $v^2/2$ is dissipated within the mixing time $\tau = l/v$, and that $\eta = \frac{1}{3}\rho v l$ by analogy with molecular viscosity. If we were to take the usual value $\varepsilon'' \approx v^3/l$, we would have to conclude that turbulence dies out for any possible β'. But if we further assume that $\eta \approx \rho v l$, attenuation of turbulence would occur only for $\beta' < {}^1/_2$. The value $\beta' = 0.2$ satisfies this condition as well, but with a comparatively small margin. Unfortunately, owing to the fact that fundamental relations of the theory of turbulence (which are defined only for a constant factor) were used, conclusions regarding the attenuation of turbulence in the protoplanetary cloud can be neither rigorous enough nor final. All one can say is that the foregoing argument tends to favor attenuation over persistence of turbulence.

It should be pointed out that the stability of the protoplanetary cloud is deduced on the assumption that angular momentum increases away from the center within the gravitational field of the central body, and that the stability condition (1) is met. Strictly speaking, however, one should allow for the gravity of the cloud as well. From stellar dynamics it is known (Chandrasekhar, 1942) that, in the equatorial plane of a homogeneous, highly compressed spheroid, the force of gravity near its edges (on the outside) decreases faster than R^{-2}, and that circular orbits are unstable if the eccentricity e of the spheroid's meridional cross section exceeds the critical value $e_1 = 0.834$. A similar result will be obtained in the presence of another central mass, except that e_1 will be larger. If the central mass is ten times larger than the mass of the spheroid (a permissible assumption in the case of the Sun and protoplanetary cloud), then $e_1 \approx 0.985$. But the flattening of the protoplanetary cloud was even more pronounced. The ratio of the semiaxes of the spheroid c/a can be assumed to be roughly the same as that of the

* When referring to formulas from another chapter, the number of the chapter will be indicated by the first figure and that of the formula by the second.

half-thickness of cloud to its distance from the Sun, which is taken to be about 1/30 for the gaseous component of the cloud. Then $1-e^2=c^2/a^2 \approx 10^{-3}$ and $e \approx 0.999 > e_1$. However, in order for a region of instability to exist near such a strongly compressed, nearly homogeneous spheroid, the gradient of density in this region ought to be very high. The instability criterion associated with high gradients in flattened rotating systems was given by Lindblad in the form (see Chandrasekhar, 1942)

$$\bar{\rho}-\rho \geqslant \frac{\omega_c^2}{2\pi G}, \tag{25}$$

where

$$\bar{\rho}=-\frac{1}{4\pi G}\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0}.$$

But this criterion does not give the condition for the density gradient in explicit form. The order of magnitude of the required gradient can be estimated as follows. Let a nearly constant density distribution be replaced at distance R from the center of the system by a sharply decreasing law $\rho=CR^{-n}$. Then $\Delta\rho/\rho=-n\Delta R/R$. For a homogeneous spheroid at whose boundary the density drops abruptly to zero, there will exist a zone of instability near its edge extending from a (maximum radius) to ae/e_1, i.e., having width $\Delta a=(e/e_1-1)a$. For the instability to be maintained in this zone with a gradual decrease in density, it is necessary at the very least that ρ drop to zero within the zone, i.e., $\Delta\rho \sim \rho$. Taking $\Delta R=\Delta a$, we find that

$$n \approx \frac{R}{\Delta R}=\frac{e_1}{e-e_1}. \tag{26}$$

For $e = 0.999$ and $e_1 = 0.985$, one obtains $n \approx 70$. In the zone of the giant planets, the density determined from present planet masses decreases approximately as R^{-3}. The density gradient necessary for instability is unattainable in any part in the protoplanetary cloud.

6. Influence of the magnetic field on the stability of the rotating cloud

Some idea of the magnetic field's influence on the stability of the rotating cloud can be arrived at from certain results of Chandrasekhar (1961) for the motion of fluids between revolving cylinders (Couette flow) in the cases of a magnetic field H_z parallel to the axis of rotation and H_φ along the direction of rotation. For a field H_z of infinite conductivity, the stability condition is found to be

$$I_1 \frac{\mu H^2}{4\pi\rho} > -\int_{R_1}^{R_2} \frac{d\omega^2}{dR} R^3 \xi_R^2 dR. \tag{27}$$

This result is somewhat unexpected, since the above does not reduce to Rayleigh's criterion when $H \to 0$. For a vanishingly small field when ω is a monotonic function of R, the necessary and sufficient condition for instability is that ω increase with R. In the protoplanetary cloud ω decreases with R

and therefore for a weak magnetic field the cloud should be less stable than we found earlier in the absence of the field, when for its stability it was sufficient that ωR^2 increase. However, this result is not confirmed when the premises are more general and allowance is made for the dissipative properties of the medium — the viscosity ν is not zero and the electrical conductivity σ is not infinite. In the case of the magnetic field H, lying along the axis of rotation, taking the distance between the inner and outer walls of the cylinder ($d=R_2-R_1$) to be small compared with R and writing the angular velocity of rotation in the form $\omega=A+B/R^2$, Chandrasekhar obtained a theoretical expression for the dependence of Taylor's critical number $\hat{T}_c$ on the dimensionless parameter $Q=\mu_* H^2 d^2/4\pi\rho\nu\eta$, where $\eta=1/4\pi\mu_*\sigma$; μ_* is the magnetic permeability.

The dependence of $\hat{T}_c$ on Q is almost linear. For $Q\to\infty(\sigma\to\infty$ or $\nu\to 0)$ the ratio $\hat{T}_c/Q$ tends asymptotically to the constant value N (107 for nonconducting, 451 for conducting walls). Since

$$\hat{T}=2(1+\mu)A\frac{\omega_1 d^4}{\nu^2}, \qquad A=\frac{\omega_2 R_2^2-\omega_1 R_1^2}{R_2^2-R_1^2}, \tag{28}$$

the stability condition $\hat{T}<\hat{T}_c$ can be written as

$$\mu_* H^2>-\frac{2(1+\mu)\rho d^2\omega_1(\omega_2 R_2^2-\omega_1 R_1^2)}{\mu_*\sigma\nu N(R_2^2-R_1^2)}. \tag{29}$$

Hence for $H\to 0$ the stability condition reduces to Rayleigh's criterion: increase of ωR^2 with R. The presence of a magnetic field increases stability.

In a system with differential rotation, the toroidal field H_φ is more probable. Then the stability condition for $\sigma=\infty$ and $\nu=0$ has the form

$$\frac{d}{dR}(\omega R^2)^2-\frac{\mu}{4\pi\rho}R\frac{d}{dR}\left(\frac{H_\varphi}{R}\right)^2>0. \tag{30}$$

For $H_\varphi\to 0$ this condition reduces to Rayleigh's criterion. If H_φ increases more slowly than R, the magnetic field will increase the stability of the rotating cloud. But even a field rapidly increasing with R would be unable to neutralize the stabilizing effect of rotation if its strength inside the cloud was less than 10 oersted, and the cloud would continue to remain stable. The protoplanetary cloud probably had a toroidal field which grew weaker away from the Sun. The presence of such a magnetic field could only have contributed to the stability of the cloud as obtained above without allowance for a field.

Chapter 3

FORMATION OF THE DUST LAYER

7. Barometric formula for flat rotating systems

We shall say that a system is flat if its thickness H is much less than the distance R from the center of the system. The protoplanetary cloud belongs to this category. Its thickness is determined by the thermal velocities of the particles and can be obtained from an expression similar to the barometric formula for the Earth's atmosphere. Let us assume that the cloud consists of a one-component gas in laminar rotation. Its equilibrium in the radial direction (perpendicular to the axis of rotation) will be maintained mainly by the rotation. The gas pressure gradient along R will be very low (von Weizsäcker, 1944), and the rotation almost Keplerian, i. e., the force of gravity is balanced by the centrifugal force. By contrast, equilibrium in the z direction (perpendicular to the central plane) is maintained by the pressure gradient

$$\frac{dp}{dz} = \rho Z, \tag{1}$$

where Z is the acceleration of gravity in the z direction. It is due to the Sun's and the cloud's gravitation. The latter is not important and can be disregarded provided the cloud's density is several times less than the critical value for which gravitational instability appears inside the cloud (see Section 16). Then

$$Z \approx -\frac{GM_{\odot}z}{(R^2 + r^2)^{3/2}} \approx -\frac{GM_{\odot}z}{R^3} = -\omega^2 z, \tag{2}$$

where R is the distance from the axis of rotation. Assuming that the mean velocities of the particles do not depend on z (identical particles and constant temperature) we obtain

$$\frac{1}{\rho}\frac{dp}{dz} = \frac{1}{3}\overline{v^2}\frac{1}{\rho}\frac{d\rho}{dz} = -\omega^2 z \tag{3}$$

and

$$\rho(z) = \rho_0 e^{-\frac{3\omega^2}{2}\frac{z^2}{\overline{v^2}}}. \tag{4}$$

Consequently, the thickness H of the homogeneous layer is

$$H \equiv \frac{\sigma}{\rho_0} = \frac{1}{\rho_0}\int_{-\infty}^{+\infty}\rho dz = \frac{1}{\omega}\sqrt{\frac{2\pi}{3}\overline{v^2}} = \frac{\pi}{2\omega}\bar{v} = \frac{P}{4}\bar{v}, \tag{5}$$

where we take $\overline{v^2}=\frac{3\pi}{8}v^2$, which holds for a Maxwellian velocity distribution.

8. Flattening of the dust layer in a quiescent gas

In Chapter 2 we saw that chaotic macroscopic motions arising during the formation of the gas-dust cloud enveloping the Sun were rapidly damped and that the rotation of the cloud tended to become laminar. The fact that the chemical composition of the planets differs from that of the Sun (i. e., from the assumed primordial composition of the cloud) indicates that the density of the gaseous component of the cloud was not so high as to lead to gravitational instability and the resulting formation of stable gaseous clusters. The cloud's subsequent evolution must therefore have been linked to the presence of a dust component.

Once the turbulent motions in the gas had been damped, solid particles began to settle on the central plane. The settling time can be estimated from the equation for the motion of a particle along the z axis. For constant particle mass it has the form (Safronov, Ruskol, 1957)

$$\frac{d^2z}{dt^2}+a'\frac{dz}{dt}+\omega^2z=0, \tag{6}$$

where

$$a'=\frac{\rho_g v_g}{r\delta}; \tag{7}$$

r and δ are respectively the radius and density of the particle, and ρ_g and v_g the gas density and thermal velocities of the molecules. Larger particles with radii $r>\frac{\rho_g v_g}{2\delta\omega}$ describe attenuating oscillations with respect to the plane $z=0$. Smaller particles sink asymptotically toward the plane $z=0$. Their z coordinate decreases e times within the time

$$t=\frac{\rho_g v_g}{r\delta\omega^2}, \tag{8}$$

which amounts to about $3\cdot10^5$ turns around the Sun for particles of radius 10^{-4} cm.

Particle aggregation during collisions contributes considerably to the speed of settling. Consider the settling of the larger particles, assuming for simplicity that all others are immobile. Let the particle m absorb all other particles it encounters on its way to the plane $z=0$. Its mass increment will be determined by the distance traveled:

$$dm=4\pi r^2\delta dr=-\pi r^2\rho_p dz. \tag{9}$$

From this we obtain the expression for the radius of the sinking particle:

$$dr=-\frac{\rho_p dz}{4\delta};\qquad r=r_1+\frac{\rho_p}{4\delta}(z_1-z). \tag{10}$$

For a particle of variable mass m, equation (6) is replaced by

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right)+ma'\frac{dz}{dt}+m\omega^2z=0, \tag{11}$$

or

$$\ddot{z} + a'\left(1 - \frac{3\rho_p \cdot \dot{z}}{4\rho_g \cdot v_g}\right)\dot{z} + \omega^2 z = 0, \tag{12}$$

where coefficient a' is already dependent on z.

For small particles and z not small, the first term $\ddot{z}$ is very small, compared with the others* and can be disregarded. Furthermore, the second term in brackets is small compared with unity. Therefore instead of (12) we can write

$$\frac{\dot{z}}{z} \approx -\frac{\omega^2}{a'} = -\frac{\omega^2\delta}{\rho_g \cdot v_g}\left[r_1 + \frac{\rho_p}{4\delta}(z_1 - z)\right] = -(c_1 - c_2 z). \tag{13}$$

Integrating, we obtain the time in which the particle sinks from z_1 to z:

$$t(z_1, z) \approx \frac{1}{c_1}\ln\left(\frac{z_1}{z}\,\frac{1 - c_2 z}{1 - c_2 z_1}\right). \tag{14}$$

The approximate expression (13) is not suitable for small z. Setting $z \sim z_1/2$, we find that, even for very small particles with $r_1 \sim 10^{-5}$ cm, the time for the particles to settle to the central plane of the cloud when allowance is made for their growth will amount to only about 10^3 revolutions of the cloud around the Sun. In this time the particle radius will increase by

$$\Delta r = \frac{\rho_p z_1}{4\delta} \leqslant \frac{\sigma_p}{8\delta}. \tag{15}$$

At the Earth's distance from the Sun $\sigma_p \approx 10$ g/cm^2 and $\Delta r \sim 1$ cm.

Thus in the absence of turbulence, solid particles settle down to the equatorial plane within a very short time, forming there a flat dust layer of high density. When the density of this layer becomes critical gravitational instability develops and numerous dust condensations are formed (see Chapters 5 and 6).

9. Thickness of the dust layer in turbulent gas

The dust particles which formed as a result of the condensation of non-volatile substances in the cloud originally traveled together with the gas. Amounting to only 1% of the cloud mass, they had little influence on the character of the random motion of the gas. As particle sizes increased and random velocities in the gas decreased, the particles began to sink to the central plane of the cloud. In the inner part of the cloud nearer to the Sun, attenuation of random motions may have been less than complete thanks to the perturbing effect of solar activity (corpuscular fluxes, magnetic perturbations, etc.). For brevity we shall call such motions turbulent, not investing the term with the rigorous meaning it has in the theory of turbulence. These motions in the gas determined the relative velocities of the solid particles and consequently the thickness of the dust layer. We will first evaluate the relative velocities of the particles, making use of the concepts of the semiempirical theory of turbulence.

* The time in which the particle velocity tends asymptotically to the value $\dot{z}$, which is obtained from (12) for $\ddot{z} = 0$, is of the order of $1/a'$, i.e., about $r \cdot 10^{-3}$ turns. In the first phase of settling, the relative variation of z in this time is small.

Let v be the mean turbulent velocity in the gas, τ the mixing time, i.e., the time within which the turbulent eddy mixes with the surrounding medium and ρ_g and v_g the density of the gas and the thermal velocity of the molecules, respectively. The mean acceleration of the volume elements of the gas is given by

$$g_t \sim \frac{v}{\tau}. \tag{16}$$

The gas carries solid particles along with itself. But its motion is not transmitted to the particles completely. Let Δv be the rate at which particles of mass m, radius r and density δ lag behind the gas. Then the particle acceleration can be taken as

$$g_p \approx \left(1 - \frac{\Delta v}{v}\right) g_t. \tag{17}$$

The particle acquires this acceleration under the influence of the gas pressure, and for a rarefied gas it can be written as

$$g_p = \frac{F}{m} = \frac{4\pi}{3} \frac{\rho_g v_g r^2 \Delta v}{\frac{4\pi}{3} \delta r^3} = \frac{\rho_g v_g}{\delta r} \Delta v. \tag{18}$$

Setting (17) and (18) equal, we obtain the relation between the particle radius and its rate of lag with respect to the gas, Δv:

$$r \approx \frac{\rho_g v_g \tau}{\delta \left(1 - \frac{\Delta v}{v}\right)} \frac{\Delta v}{v}. \tag{19}$$

The particles' separation from the gas becomes significant for $\Delta v \geqslant v/2$, i.e., for

$$r \geqslant r_0 = \frac{\rho_g v_g \tau}{\delta}. \tag{20}$$

The relative particle velocities v_p inside the cloud can be assumed to be $v - \Delta v$. Then from (19) and (20)

$$v_p = v - \Delta v = v \cdot \frac{r_0}{r_0 + r}. \tag{21}$$

Urey concluded that dust is important in the attenuation of turbulence (1958). He calculated the Reynolds number for a cloud a quarter of whose mass consists of solid particles, taking the mean free path of particles between mutual collisions as the characteristic dimension l and $l dV_c/dR$ (i.e., variation of the circular velocity along the path l) as characteristic velocity. Urey obtained Re $\approx$ 70 for a Roche density* and the particle radius

* The concept of Roche density is directly related to the more familiar concept of the Roche boundary, which characterizes the distance at which a fluid body of density ρ moving around a central body of density ρ_0 and radius R_0 will disintegrate under the influence of its tidal forces: $R = 2.455 R_0 \sqrt[3]{\rho_0/\rho}$. Hence the expression for the critical density ρ_R at which disintegration of the body occurs:

$$\rho_R = 14.8 \rho_0 (R_0/R)^3 = 14.8 \rho^*,$$

where

$$\rho^* = 3M_0/4\pi R^3 = \rho_0 (R_0/R)^3.$$

$r = 1$ cm at the Earth's distance from the Sun. But no allowance was made in this calculation for the fact that the dust grains are being carried off by the gas and therefore do not reduce the mixing length l to the atomic mean free path among the grains. If the particles are all the same size their total resistance per cm^3 to the gas, according to (18)—(20), will be given by

$$Fn_p = F\frac{\rho_p}{m} = \frac{\rho_p r_0}{\tau}\frac{\Delta v}{r} = \frac{\rho_p v}{\tau}\frac{r_0}{r_0 + r}. \tag{22}$$

For a constant density ρ_p of the solid matter, as the particles shrink their resistance increases, tending to a limit which is only twice as large as the resistance for $r = r_0$. Incidentally, the Reynolds number which Urey obtained is proportional to r^3. If we were to perform the substitution $r = r_0$ (Urey takes $\rho = 10^{-6}$ g/cm^3 and $\delta = 0.07$ g/cm^3 for solid hydrogen, with $r_0 \approx 70$ km), we would obtain Re $\sim 10^{14}$.

The above relation (22) enables the attenuation of turbulent motions by solid particles to be estimated. Particle acceleration by the gas g_p is accompanied by a corresponding deceleration of the gas by the particles:

$$\frac{dv}{dt} = -\frac{\rho_p}{\rho_g} g_p = -\frac{\rho_p}{\rho_g}\frac{F}{m}. \tag{23}$$

From (22) the characteristic time τ_p of damping of turbulence due to the particles is given by

$$\tau_p \equiv -\frac{v}{dv/dt} = \frac{r_0 + r}{r_0}\frac{\rho_g}{\rho_p}\tau. \tag{24}$$

Consequently attenuation of turbulence by solid particles becomes substantial only when $\rho_p \approx \rho_g$.

The thickness of the dust layer inside the gas is determined in the same way as the uniform height of a heavier component of the gaseous mixture. In the absence of macroscopic motions inside the gas it is uniquely determined by the thermal velocities of the particles according to (5). But if the gas is being vigorously mixed (e.g., due to convection), its components will not be separate and will have the same uniform height. Hence the uniform thickness H_p of the dust layer in a turbulent gas lies between its minimum H_{pm}, given by the barometric formula (5) for the particle velocity v_p according to (21), and the uniform thickness H_g of the gas, obtained from (5) for $v = v_g$:

$$H_{pm} = \frac{\pi v_p}{2\omega}, \qquad H_g = \frac{\pi v_g}{2\omega}. \tag{25}$$

The upper limit of thickness of the dust layer can be substantially reduced. The "Stokes" velocity v_s of particles sinking in the gas due to the acceleration Z caused by the Sun's gravity can be found from (18) if we take $g_p = Z$:

$$v_s = Z\frac{\delta r}{\rho_g v_g} = \omega^2 z\frac{\delta r}{\rho_g v_g} = \omega^2 z\tau\frac{r}{r_0}. \tag{26}$$

As long as v_s remains greater than the turbulent velocity v in the gas, the particle will drop continuously to the central plane. Therefore the half-thickness of the homogeneous layer ($H_p/2$) is less than the z determined from (26) for $v_s = v$. In view of (5) and (21) we obtain

$$H_p < 2z = \frac{2v}{\omega^2\tau}\frac{r_0}{r} = \frac{4}{\pi\omega\tau}\frac{r_0+r}{r}H_{pm} = \frac{4}{\pi\omega\tau}\frac{v}{v_g}\frac{r}{r_0}H_g. \tag{27}$$

All particles with $r > \frac{4}{\pi\omega\tau}\frac{v}{v_g}r_0$ must move toward the central plane, and the larger they are the flatter the layer they will form there. The thickness of the layer lies within the limits

$$H_{pm} < H_p < \frac{4}{\pi\omega\tau}\frac{r_0+r}{r}H_{pm}. \tag{28}$$

In determining the scale of turbulence in a medium with differential rotation from Heisenberg's theory, Chandrasekhar and Ter Haar (1950) assume

$$l \propto R, \qquad v \propto V_g, \qquad \tau \sim 1/\omega. \tag{29}$$

It seems, therefore, that one could set $\omega\tau \approx 1$ in all the preceding relations. Thus sufficiently large particles ($r > r_0$) characterized by a relative independence of motion will lie in a layer of thickness $H_p \approx H_{pm}$.

In order for the critical density to be reached in the dust layer, the latter must achieve a very high degree of quiescence and flattening. According to Ruskol (1960), after allowing for the gravitation of Sun and cloud the density ρ_0 in the cloud's central plane and its surface density σ are related by

$$\sigma = \sqrt{\frac{2\mathfrak{R}T\rho_0}{\pi G\mu}}\,\mathfrak{I}(\eta), \tag{30}$$

where $\eta = \rho_0/\rho^*$; $\rho^* = 3M_\odot/4\pi R^3$. The value of $\mathfrak{I}(\eta)$ is close to unity for a density of the order of the Roche density; for $\rho_0 = 2\rho^*$, $\mathfrak{I} = 0.9$.

For the dust layer it is necessary to set $\sigma = \sigma_p$ in (30). Taking $\mathfrak{R}T/\mu = \overline{v_p^2}/3$, we obtain

$$\overline{v_p^2} = \frac{3\pi G\sigma_p^2}{2\mathfrak{I}^2\rho_0}. \tag{31}$$

For gravitational instability in the terrestrial zone it is necessary that $\rho_0 \sim 3 \cdot 10^{-7}$ (see Chapter 5). From (31) we find that for $\sigma_p = 10$ the particle velocity $v_p \sim 11$ cm/sec. The layer's thickness $H = \sigma/\rho_0$ must also be very small: $H/R \approx 2 \cdot 10^{-6}$. In the region of planetary giants conditions were more favorable for gravitational instability; in the Jupiter zone one must have $v_p \sim 270$ cm/sec and $H/R \sim 10^{-4}$. The perturbations caused by solar activity were more effective in the inner parts of the cloud and the associated random velocities increased toward the Sun. Conditions were therefore particularly unfavorable for gravitational instability of the dust layer within the region of inner planets.

The size of bodies capable of separating out of the gas and forming a flattened layer increases as random motions in the gas grow stronger. But with increasing size the gravitational interaction of the bodies and the relative velocities this interaction produces also increase. According to (7.12), in a system of identical bodies of mass m and radius r, the relative velocities are given by $\sqrt{\frac{Gm}{\theta r}}$. For $\theta \approx 3$, bodies with $r \approx 2 \cdot 10^4$ cm have a

velocity of 11 cm/sec. On the other hand, from (21) we find that such bodies will have the velocity $v_p = 11$ cm/sec when $v \approx 380$ cm/sec. Thus turbulent velocities in the gas should be less than 380 cm/sec to achieve gravitational instability in the solid body layer within the Earth zone. Otherwise, due to the increased gravitational interaction of the growing bodies, flattening will give way to swelling before the critical density of the layer is reached.

Table 3 gives the limiting values of the turbulent velocity v_M in the gas at various distances from the Sun. When the velocity of random motions in the gas $v > v_M$, gravitational instability could not have arisen in the dust layer.

TABLE 3

	Mercury	Earth	Jupiter	Neptune
σ_p, g/cm^2	1.5	10	20	0.3
σ_g/σ_p	300	200	70	70
v_{cr}, cm/sec . . .	0.4	11	270	50
r_M, cm	$7 \cdot 10^2$	$2 \cdot 10^4$	$6 \cdot 10^5$	10^5
v_M, cm/sec . . .	4	380	$4 \cdot 10^5$	10^6

The values of v_{cr} in the third row were obtained from formula (31) and represent the velocities of bodies at which the layer produced by them becomes gravitationally unstable ($\rho_0 = 2.1\rho^*$). The values of r_M in the next row represent the radii of bodies whose velocities equal v_{cr} due to their gravitational interaction. The values of v_M are turbulent velocities within the gas obtained from (21) for $v_p = v_{cr}$ and $r = r_M$.* We see from the table that the v_M are very small for the planets of the Earth group and especially for the Mercury zone, whose proximity to the Sun — a source of various perturbations — makes it practically impossible to reach such small v_M.

Thus it seems highly probable that gravitational instability of the dust layer was present in the zone of planetary giants but not in the Mercury zone. The influence of random motions of the gas on the solid material was substantial only, so it seems, among the innermost planets (Mercury and possibly Venus), within range of the perturbing effect of solar activity. Where gravitational instability could not have arisen, growth of the bodies must have been due to their aggregation in collisions.

* For the motion of bodies in a gas the parameter θ is several times greater than the value $\theta = 3$ adopted above (see Table 11 in Chapter 7). Consequently, the values of r_M and v_M should be greater than given in Table 3 (about 2 — 3 times greater in the region of the Earth group and approximately 30% greater in the region of planetary giants).

Chapter 4

TEMPERATURE OF THE DUST LAYER

10. Statement of the problem

One of the most important characteristics of the dust layer formed in the equatorial plane of the protoplanetary cloud was its temperature, for on this depended the chemical composition and mass of the layer. The chemical composition of the dust layer largely determined the chemical composition of the planets; the mass of the layer determined the size and mass of the condensations formed inside it. Differences in temperature conditions account for the division of the planets into two groups. It has been conjectured (e.g., by Urey, et al.) that condensation of hydrogen could have occurred in the large-planet region. It is natural to suppose that the element most abundant in the cosmos should have been a major component originally of the protoplanetary cloud. When studying the cloud's evolution it is therefore particularly important to establish whether hydrogen could have condensed inside it to the solid state.

Basing himself on the theory of common formation, Schatzman (1960) considered the warming of the cloud by cosmic rays emanating from the Sun at the stage of gravitational contraction, a period of intense electromagnetic activity (for a solar radius twice as large as today). The turbulent magnetic field enveloping the Sun prevented the rapid escape of cosmic rays from the vicinity, and a large fraction of these rays was absorbed by the particles of the protoplanetary cloud. For a total flux of cosmic rays of 10^{33} erg/sec, the temperature of the cloud was of the order of tens or hundreds of degrees Kelvin. However, the parameters used are highly indeterminate.

Gurevich and Lebedinskii (1950) obtained the temperature distribution in a uniform, optically thick two-dimensional layer extending in the direction R and of constant thickness H along z, which is being warmed by ordinary solar radiation for $R = 0$ and is emitting in the z-direction. The temperature of the layer decreases exponentially with R according to the law exp $(-R/4H)$, and is very low at distances R many times greater than H from the source of heat. The radiation, propagating inside the layer by diffusion, easily escapes from the layer in the z-direction, and only a negligible fraction penetrates to great distances R (Figure 1).

However, the dust layer revolving around the Sun in its gravitational field was not plane-parallel or homogeneous. Near the Sun its thickness was substantially less than at a distance, and its density decreased rapidly with z. The Sun lay largely outside the layer, and its radiation, propagating almost parallel to the layer, penetrated into the upper rarefied regions to great distances, falling into the layer after scattering in these regions. Although

the scattered radiation did not amount to a great deal, it was sufficient to prevent the temperature of the layer from dropping to extremely low values (Safronov, 1962b).

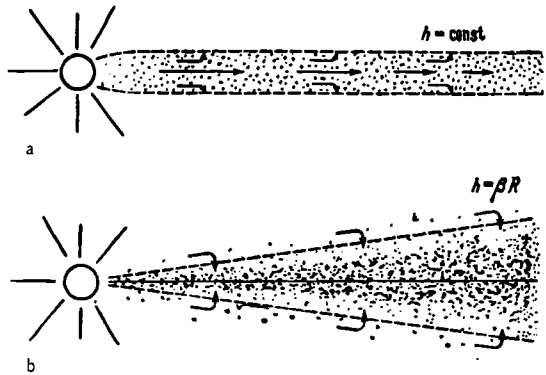

FIGURE 1. Warming of dust layer by solar radiation, according to Gurevich and Lebedinskii (a), and warming of the layer due to radiation by scattered particles lying in rarefied part of layer (b).

For a numerical estimate of the warming due to solar radiation scattered in the rarefied part of the layer, it is necessary to devise a reasonable model. The absence of a clear idea of the genesis of the protoplanetary cloud makes this task difficult. In Chapter 1 we noted that the different sections of the cloud need not have evolved simultaneously. The inner parts of the layer evolved much faster than the outer ones, but they could have been formed later than the outer sections, as in Schatzman's theory. The role of solar activity, which slowed down the flattening of the layer, and that of the magnetic field, are also unclear. As these questions are undecided we will consider the simplest model, a single optically thick dust layer having the same coefficient of opacity $\varkappa$ (per unit mass) at all distances from the Sun. We assume the same intensity of solar radiation as today. Owing to the Sun's heightened luminosity in the gravitational contraction phase, the temperature of the dust layer in this phase must have been correspondingly higher. By temperature of the layer we will understand the temperature of a black body (black ball) placed at the given point, which is uniquely determined by the mean intensity J of the integral (over all wavelengths) radiation at this point. The temperature of real particles may be different. This applies in particular to particles at the surface of the layer which absorb shortwave solar radiation in the visible region of the spectrum and emit in the far infrared. However, in the central portion of the dust layer nearly all the radiation is longwave as it has undergone repeated absorption and reemission by the particles. Here it is possible to have local thermodynamic equilibrium in which the temperature of the particles is almost identical with the black-body temperature.

Essentially the problem breaks down into two parts: determining the temperature distribution inside the layer for a specified value at its boundary; and determining the boundary value. The first is relatively simple

to solve since the thickness of the layer is much smaller than the distance from the Sun. As a result the temperature inside the layer is nearly invariant with z (see Section 11). In practice, therefore, the problem reduces to finding the temperature at the boundary of the layer. Its density decreases indefinitely with z, and the concept of a boundary z_1 is arbitrary. Whereas the position of the "surface" of the layer may be defined as the smallest z for which $\tau(z) \approx 0$, the position of the "boundary" z_1 of the layer must satisfy two requirements. First, in order for the equation used in Section 11 for the stream of radiation to be valid up to z_1, the mean free path of the quanta must be considerably less than the half-thickness h of the layer. This condition is met when the optical thickness $\tau(z_1)$, reckoned inward from the surface, amounts to a few units. Second, outside the layer and at its surface there is direct solar radiation and the black-body temperature is higher than inside the layer. It decreases inward, rapidly approaching its limiting value. This is the value which should be used for $T(z_1)$. In practice this value is reached when $\tau(z_1)$ is also of the order of a few units. Quantity $T(z)$ in Section 12 is obtained assuming gray absorption in the layer, i.e., it is assumed that complete absorption (true absorption plus scattering) is independent of wavelength. Then the mean free path will be the same for all quanta, and when evaluating radiative transfer one can therefore consider integral (over the entire spectrum) rather than monochromatic radiation. For integral radiation, by contrast with monochromatic radiation, radiative equilibrium exists since the radiation absorbed by a particle is reemitted in the longwave region of the spectrum instead of fading away. Consequently from the energy standpoint, in gray absorption light propagation in the medium takes place in the same way as in pure scattering. This makes it possible to use the results of the theory of diffuse reflection and transmission of the light incident at the boundary of a plane-parallel atmosphere. In the case of isotropic elastic scattering (no absorption), as one moves down through the atmosphere the mean radiation intensity will tend to a definite limit which depends on the intensity of the incident radiation and on the angle of its incidence. In the case of gray absorption it is the mean intensity of integral radiation which must tend to this limit.

11. Temperature distribution inside the dust layer

We will consider first a plane-parallel dust layer having a plane of symmetry $z = 0$ and optical properties dependent only on the z-coordinate. The integral radiation flux E inside the layer obeys the continuity equation, which in the cylindrical coordinate system (R, z) has the form

$$\operatorname{div} E = \frac{\partial E_R}{\partial R} + \frac{1}{R} E_R + \frac{\partial E_z}{\partial z} = 0. \qquad (1)$$

Here E_R and E_z are the flux in the directions R and z, respectively. The right-hand side is zero because there are no sources of energy in the layer and the radiation, having undergone "true absorption," is again reemitted in other frequencies. Consider the case of gray absorption and isotropic reemission. The relation between the integral flux (over all wavelengths)

and the integral mean intensity of radiation $J=\frac{1}{4\pi}\int I d\omega$ can then be determined directly from the diffusion equation

$$E_R=-\frac{4\pi}{3\alpha}\frac{\partial J}{\partial R},$$
$$E_z=-\frac{4\pi}{3\alpha}\frac{\partial J}{\partial z}, \tag{2}$$

where $\alpha=\varkappa\rho$ is the coefficient of absorption per unit volume. In the case of particles of the same size, $\varkappa$ will be independent of z and $\alpha \propto \varrho$. If, moreover, the kinetic temperature does not vary with z, then $\rho \propto e^{-z^2/h^2}$. However, in the case of particles of varying sizes the smaller ones settle down more slowly to the central plane; $\varkappa$ will then increase with z (it is assumed that the particle diameter is greater than the wavelength), and ρ will decrease more slowly than e^{-z^2/h^2}. To simplify the calculations we can take

$$\alpha=\alpha_0 e^{-z/h} \tag{3}$$

and h = const. Then from (1) and (2) we obtain, for $z \geqslant 0$,

$$\frac{\partial^2 J}{\partial R^2}+\frac{1}{R}\frac{\partial J}{\partial R}+\frac{\partial^2 J}{\partial z^2}+\frac{1}{h}\frac{\partial J}{\partial z}=0. \tag{4}$$

The boundary conditions will be as follows:

$$J=J_0 R^{-p} \text{ for } z=z_1, \qquad \frac{\partial J}{\partial z}=0 \quad \text{for } z=0. \tag{5}$$

An approximate solution of equation (4) can be found for the layer with $h \ll R$ (and correspondingly $z \ll R$). It is natural to expect that the value of J inside the layer will not differ much from its value (5) at the boundary. Let us write it as follows:

$$J=\frac{J_0}{R^p}\left[1+\frac{u_1(z)}{R^2}+\frac{u_2(z)}{R^4}+\ldots\right]. \tag{6}$$

Inserting this expression into (4) and equating the coefficients of different powers of R to zero, we obtain equations for $u_i(z)$. Solving the latter and choosing constants of integration such that the boundary conditions (5) are fulfilled, we obtain

$$u_1(z)=hp^2[z_1-z-h(e^{-z/h}-e^{-z_1/h})];$$
$$u_2(z)=h^2p^2(p+2)^2[(z_1+h+he^{-z_1/h})(z_1-z)-\frac{1}{2}(z_1^2-z^2)-$$
$$-h(z_1+h+he^{-z_1/h})(e^{-z/h}-e^{-z_1/h})-h(ze^{-z/h}-z_1e^{-z_1/h})]. \tag{7}$$

The substitution shows that the odd powers in square brackets in (6) will drop out.

The quantity z_1 depends on $\varkappa$ and roughly equals two to three half-thicknesses h. For $z_1/R \sim 10^{-2}$ we have $u_1(z) R^{-2} < 10^{-3}$ and $u_2(z) R^{-4} \sim 10^{-8}$. Thus in the solution of (6) the second term plays an insignificant role while the third term is negligible. The series converges very rapidly and the z-dependence of J is determined practically only by the term with $u_1(z)$. The mean radiation intensity J increases very slowly from the boundary of the layer to the central plane. This increase is due to the fact that the radiation reaching the central plane comes from a region at the boundary of dimensions $\sim z_1$. Its intensity varies with R according to (5). The mean value R^{-p} within such a radius around a point situated distance R_0 from the Sun will exceed the value R_0^{-p} by $\sim z_1^2/R_0^2$. Quantity J is related to the black-body temperature T in the layer by the simple relation

$$T^4 = \frac{\pi}{\sigma'} J, \tag{8}$$

where σ' is the Stefan-Boltzmann constant. Thus we can assume that the temperature of the dust layer is nearly the same throughout its thickness and that it depends only on R.

In the following section we will adopt a more precise model of the layer in which the density ρ_0 is a function of R and $h = \beta R$. However, our conclusion regarding the very weak dependence of J on z still holds. Indeed, if we take $\alpha = \alpha_0 R^{-n} e^{-z/h}$, the constant factor $(1+n)$ will appear in the second term of equation (4). The approximate solution of this new equation will be found in the same way as above. It differs from (6) in the appearance of an additional factor $1 - n/p$ in $u_1(z)$, which even reduces $u_1(z)$ to some extent. If one further takes $h = \beta R$, another additional factor $(1 - z/h)$ appears in the second term of (4). The expression for $u_1(z)$ becomes more complex, but the order of magnitude remains as before.

12. Temperature of the layer near the surface

The theory of diffuse reflection and transmission of light incident upon the boundary of a plane-parallel atmosphere, developed by Ambartsumyan (1942), Sobolev (1956), Chandrasekhar (1950) and others, enables us to determine the density of radiation at large optical thicknesses as a function of the intensity of the incident light and the angle of incidence. For isotropic elastic scattering of uniform radiation incident at angle θ to the inner normal, the mean radiation intensity $J_\nu(\tau)$ will tend with increasing optical thickness τ (reckoned from the surface to the interior of the layer) to the following finite limit:

$$J_\nu(\infty, \mu) = \frac{\sqrt{3}}{4\pi} E_\nu(\mu) \varphi(\mu), \tag{9}$$

where $\mu = \cos\theta$; $E_\nu(\mu)$ is the stream of energy of frequency ν incident on the surface per square centimeter per second in the direction μ, and $\varphi(\mu)$ is a function given by the integral equation

$$\varphi(\mu) = 1 + \frac{1}{2}\mu\varphi(\mu) \int_0^1 \frac{\varphi(\mu')}{\mu + \mu'} d\mu'.$$

Tables of numerical values of this function computed by the method of successive approximations are given by Sobolev and Chandrasekhar. For isotropic scattering the function $\varphi(\mu)$ is nearly linear: $\varphi(\mu) \approx 1 + 2\mu$. The mean and maximum errors in this approximation are respectively about 2% and less than 4%.

The relation (9) is valid for monochromatic radiation in any frequency ν. Since the black-body temperature of the layer is determined by the integral radiation density in all wavelengths, it is sufficient to evaluate the mean integral intensity $J(\infty, \mu)$, without calculating $J_\nu(\infty, \mu)$. Since (9) does not contain the coefficient of absorption it is obviously valid for integral radiation as well:

$$J(\infty, \mu) = \frac{\sqrt{3}}{4\pi} E(\mu) \varphi(\mu), \tag{9'}$$

where

$$E = \int_0^\infty E_\nu d\nu, \qquad J = \int_0^\infty J_\nu d\nu.$$

Whereas relation (9) for $J_\nu(\infty, \mu)$ holds only for pure scattering, relation (9') for $J(\infty, \mu)$ is valid also for gray absorption. Indeed, if the total absorption coefficient is independent of wavelength, the equation of transfer will be the same for integral and monochromatic radiation. For integral radiation radiative equilibrium will also obtain, since one is dealing with a stationary case and the conversion of radiant energy into other forms of energy is not assumed. Both scattering and reemission are assumed to be isotropic.

In contrast with the semi-infinite atmosphere, a flat layer is symmetrically illuminated on both sides. But provided the optical thickness of the layer is large enough, for the same intensity of incident radiation the radiation density in its central plane will be the same as in a semi-infinite atmosphere at large τ, i.e., it will be given by (9'). In both cases the radiation flux across any small area parallel to the layer will be zero, as the amount of radiation reflected by the surface is equal to the amount incident upon it. A layer irradiated from both sides receives twice the amount of radiation, but its surface is also twice as large. That is, one square centimeter of its surface reflects as much as one square centimeter of the semi-infinite atmosphere.

The half-thickness h of the dust layer is small compared with the distance R from the Sun. It is determined by the relative velocities of the particles and depends on R. The simplest and at the same time most realistic assumption is that $h \propto R$. This corresponds to a relative particle velocity proportional to the circular velocity (see (3.5)) and to a kinetic temperature $\propto R^{-1}$.

If the layer is very thin and $h \ll R_\odot$, one may disregard all effects caused by the departure of the real cloud and incident radiation from the ideal model for which relations (9) and (9') are valid. The calculation is then particularly simple. Since the Sun's radius $R_\odot \ll R$, we have

$$\mu = \cos\theta \leqslant R_\odot/R \ll 1 \quad \text{and} \quad \overline{\varphi(\mu)} \approx 1 + R_\odot/R \approx 1.$$

Therefore according to (9'), to evaluate the temperature of the layer it is sufficient to find the flux of solar radiation across one square centimeter of the surface of the layer. The element of area $ds = 2\sqrt{R_\odot^2 - \zeta^2}d\zeta$ of the solar disk situated at height ζ will give the flux

$$dE = \frac{L}{4\pi R^2}\cos\theta\frac{ds}{\pi R_\odot^2} = \sigma' T_*^4\left(\frac{R_\odot}{R}\right)^2\frac{\zeta}{R}\frac{2\sqrt{R_\odot^2 - \zeta^2}d\zeta}{\pi R_\odot^2},$$

where L is the Sun's luminosity, T_* its effective temperature and σ' the Stefan-Boltzmann constant. The flux from the entire solar disk is given by

$$E = \sigma' T_*^4\frac{1}{\pi R^3}\int_0^{R_\odot} 2\sqrt{R_\odot^2 - \zeta^2}\,\zeta d\zeta = \frac{2}{3\pi}\left(\frac{R_\odot}{R}\right)^3\sigma' T_*^4. \tag{10}$$

From (8) and (9'), for $\varphi(\mu) = 1$ the temperature of the layer will be

$$T_0 = \left(\frac{\sqrt{3}}{4}E\right)^{1/4} = \left(\frac{1}{2\sqrt{3}\pi}\right)^{1/4}\left(\frac{R_\odot}{R}\right)^{3/4}T_*. \tag{11}$$

However, if h is of the same order as $R_\odot$ or larger, such a calculation will not suffice. The radiation incident upon the layer, propagating nearly parallel to its surface, travels a considerable distance within the outer, rarefied portion of the layer. The fraction of radiation which reaches any given point will depend to a large extent on the density distribution of the matter on the way. If the layer is not plane-parallel and not uniform with respect to R, the temperature must be calculated on the basis of a concrete model. It should also be recalled that the Sun is not an infinitely distant source. The optical thickness $\tau(\theta)$ for θ close to $\pi/2$ is smaller than $\tau(0)\sec\theta = \tau(0)/\mu$ for an infinitely distant source, and it does not tend to infinity for $\mu \to 0$. Quantity $J(\infty, \mu)$ is greater in this case than given by (9'), in which $E(\mu) = E'\mu$ tends to zero for $\mu \to 0$ (E' is the radiation flux outside the layer across a perpendicular area). In the rarefied portion rays propagate rigorously parallel to its surface ($\mu = 0$); after scattering they also penetrate into the layer, whereas (9') gives $E(0) = 0$ in this case.

All these additional factors can be allowed for in determining E if one computes the amount of direct solar radiation absorbed and scattered by the particles in one square centimeter of the layer's surface — more precisely, by the particles in a cylinder of unit cross-section with axis aligned with z. Let us denote this quantity by E_0. For a plane-parallel atmosphere irradiated by homogeneous parallel radiation, the incident radiation E is everywhere the same and equals E_0. In the more complicated case we are considering, the irradiation of the given area of surface is characterized in the first approximation by the value of E_0. Since $\varphi(\mu) \approx 1$, inserting E_0 in (9') we obtain, instead of $\int E(\mu)\,d\mu = E$, $J(\infty) = \frac{\sqrt{3}}{4\pi}E_0$ and, from (8), the temperature inside the layer. There remains a certain measure of inaccuracy due to the inhomogeneity of the layer and of the radiation. In reality $J(z_1)$ is determined not only by the local value of E_0 above this point but also by its value in its vicinity, since the radiation is mixed as it penetrates deeper. The mean value in the vicinity of r deviates from the value at the point by a quantity of the order of r^2/R^2, in relative units. Qualitatively this deviation is of the same nature as the increment in $J(0)$ in the central plane $z = 0$

over its value $J(z_1)$ at the boundary (Section 11). Although r exceeds h in the rarefied region, on the whole the effect is slight, since $h^2/R^2 \ll 1$.

Let us now estimate E_0. Obviously,

$$E_0 = \int_{z_1}^{z_2} \alpha dz \int_{s_\odot} \frac{I'}{\pi R_\odot^2} e^{-\tau} ds, \tag{12}$$

where I' is the intensity of solar radiation at the point (R, z) in the absence of absorption, $\alpha = \varkappa\rho$ the absorption coefficient per unit volume and τ the optical thickness along the path from the elementary area ds on the Sun's surface to the point (R, z). Since the integrand is very small outside the interval (z_1, z_2), 0 and ∞ may be used conveniently as limits of integration.

From the considerations set forth in Section 11, we take

$$\alpha = \alpha_0 e^{-z/h}, \qquad h = \beta R. \tag{13}$$

A light ray reaching the point (R, z) from the point $(0, \zeta)$ on the Sun's surface situated at distance ζ from the layer's central plane will have the following z coordinate at the distance R' from the Sun:

$$z' = \zeta + \frac{R'}{R}(z - \zeta). \tag{14}$$

In view of the smallness of ζ and z compared with R, the distance between the points $(0, \zeta)$ and (R, z) is practically equal to R. From (13) and (14), the optical thickness along the path between these points is given by

$$\tau(\zeta, z) = \int_0^R \alpha dR' = e^{-z/h} e^{\zeta/h} \int_0^R \alpha_0 e^{-\zeta/\beta R'} dR' = \tau(\zeta, 0) e^{-z/h}. \tag{15}$$

Let us substitute this expression for $\tau(\zeta, z)$ into integral (12) and first integrate it with respect to z for constant ζ and R, taking $\tau(\zeta, z)$ as the independent variable. Since from (15)

$$e^{-z/h} dz = -\frac{h}{\tau(\zeta, 0)} d\tau(\zeta, z),$$

we obtain, dropping the prime of R',

$$\int_{z_1}^{z_2} \alpha_0 e^{-z/h} e^{-\tau(\zeta, z)} dz \approx \frac{\alpha_0 h}{\tau(\zeta, 0)} \int_0^\infty e^{-\tau} d\tau = \frac{\alpha_0 h}{e^{\zeta/h} \int_0^R \alpha_0 e^{-\zeta/\beta R} dR}. \tag{16}$$

We further introduce variables u and v in place of R and ζ:

$$v = \frac{\zeta}{R_\odot}, \quad u = \frac{\zeta}{h} = \frac{R_\odot}{\beta R} v,$$

in which case

$$\frac{ds}{\pi R_\odot^2} = \frac{2\sqrt{R_\odot^2 - \zeta^2}\, d\zeta}{\pi R_\odot^2} = \frac{2}{\pi}\sqrt{1 - v^2}\, dv, \qquad dR = -R\frac{du}{u}$$

and integral (12), in view of (16), becomes

$$E_0 = \frac{2}{\pi} I' a_0 h \int_0^1 \frac{\sqrt{1-v^2}\, dv}{e^u \int_u^\infty a_0 R e^{-u} \frac{du}{u}}. \tag{17}$$

The intensity I' of the solar radiation at distance R outside the absorbing layer can be expressed in terms of the effective solar temperature T_e or in terms of the black-body temperature T' at distance R from the Sun outside the layer:

$$I' = \sigma' T_e^4 \left(\frac{R_\odot}{R}\right)^2 = 4\sigma' T'^4.$$

Introducing the value of I' into (17), from (8) and (9) we find the following expression for the temperature under the surface of the dust layer:

$$T^4 = T'^4 \frac{2\sqrt{3}}{\pi} a_0 h \int_0^1 \frac{\sqrt{1-v^2}\, dv}{e^u \int_u^\infty a_0 R e^{-u} \frac{du}{u}}. \tag{18}$$

For $a_0 R \propto R^{-n}$ we obtain

$$T^4 = T'^4 \frac{2\sqrt{3}}{\pi} \beta^{1-n} \frac{R_\odot}{R} \int_0^1 \frac{\sqrt{1-v^2}\, v^n dv}{e^{kv} \int_{kv}^\infty e^{-u} u^{n-1} du}, \tag{19}$$

where $k = R_\odot/\beta R$. For $n = 1$ this expression is easy to integrate and yields the value

$$T = T_0 \tag{20}$$

where T_0 is given by (11) and was obtained directly from the flux of solar radiation across a small surface located in the plane $z = 0$ on the assumption that the radiation was not absorbed on the way.

From (19) it is seen that the temperature of the dust layer depends not on the absolute value of the density but only on its gradient along R. The faster the density decreases with R, the lower the temperature of the layer. For $n > 1$, $T < T_0$, while for $n < 1$, $T > T_0$. As the thickness of the dust layer decreases the difference between the values of T for different n decreases; at distances up to that of Jupiter from the Earth, for $\beta = 10^{-4}$ the difference amounts to less than two degrees when n varies from -1 to $+1$.

Since $a_0 \propto \rho_0$ and $R \propto h$, $a_0 R$ is proportional to the surface density of the solid material in the dust layer. The latter may be regarded as roughly constant up to the distance of Jupiter. Beyond the Jupiter zone, the density begins to fall off sharply with R. Let us evaluate the temperature of the dust layer for the following density distributions:

$$n=0 \quad \text{for} \quad R \leqslant R_0,$$
$$n=2 \quad \text{for} \quad R > R_0.$$

Then from (19) we obtain

$$T^4 = T'^4 \left(\frac{R_0}{R}\right)^2 \frac{2\sqrt{3}}{\pi} \beta \int_0^1 \frac{\sqrt{1-v^2}\, dv}{e^{k_0 v} E_1(k_0 v) + (k_0 v)^{-2} [1 + kv - (1 + k_0 v)\, e^{(k-k_0)v}]}, \tag{21}$$

where $E_1(x)$ is the function

$$E_1(x) = \int_x^\infty e^{-x} \frac{dx}{x}. \tag{22}$$

TABLE 4

Zone of	β				T_0
	10^{-1}	10^{-2}	10^{-3}	10^{-4}	
Mercury	186	136	119	115	115
Venus	130	91	76	72	71
Earth	107	75	61	57	56
Mars	85	58	45	42	41
Jupiter	43	28	20	16.9	16.4
Saturn	23	14.8	10.5	9.5	10.3
Uranus	11.6	7.4	5.9	5.4	6.2
Neptune	7.3	5.0	4.2	3.9	4.6

Table 4 gives the temperatures T obtained by numerical integration of (19) and (22) for R equal to the distances of the planets from the Sun and for different degrees of flattening β of the dust layer. In the low temperature region stellar radiation becomes appreciable; its density is taken to correspond to warming up to 3°K. The value $\beta = 10^{-4}$ corresponds to the density $\rho \approx \rho^* = 3\, M_\odot / 4\pi R^3$ in the Jupiter zone, i.e., to a state of the layer close to gravitational instability in this region. The last column gives the values of the temperature T_0 calculated from (11).

The table shows that for a constant surface density in the layer (up to Jupiter), $T \to T_0$ when $\beta \to 0$. But if the layer is not very thin ($h > R_\odot$), its temperature will be significantly higher than T_0. The radiation density in this part of the layer decreases faster than R^{-2} but more slowly than R^{-3}, in accordance with (10). If one takes $J \propto R^{-p}$, then when β increases from 0 to 10^{-1} the exponent p decreases from 3 to 2.2. The sharp falling-off of the surface density in the region of large planets results in a faster decrease in temperature with R. Here too, however, even for the smallest β the temperature of the dust layer is much higher than that obtained by Gurevich and Lebedinskii.

13. Warming of the layer by radiation scattered in the gaseous component of the cloud

Above we examined the warming of the dust layer due to solar radiation scattered by particles within this layer. But the dust layer, embedded inside the gaseous cloud, is also warmed by radiation scattered in the gas. The foregoing discussion still holds. The departure of molecular scattering from isotropy does not substantially affect the numerical results, the difference between the functions $\varphi(\mu)$ for Rayleigh and isotropic elastic scattering amounting to less than 3%. The thickness H of the gaseous component is considerably greater than that of the dust layer. The quantity β is larger than 10^{-2} for the gas, and the temperature of the layer, from Table 4, is considerably higher. In deriving the fundamental relations, however, the upper limit of optical thickness $\tau(\zeta, z)$ along R in (16) was taken to be infinite. In a gas $\tau(\zeta, z)$ will be much smaller than in the dust layer, and if it is small at the layer's boundary z_1, then $\int_0^{\tau(\zeta, z_1)} e^{-\tau} d\tau$ in (16) is less than unity. Also, part of the solar radiation transmitted by the gas reaches the dust layer. Thus for a dust layer surrounded by gas we obtain

$$E_0 = E_{0g}[1 - e^{-\tau(\zeta, z_1)}] + E_{0p} e^{-\tau(\zeta, z_1)}, \tag{23}$$

where E_{0g} and E_{0p} are the expressions for E_0 in the form (17) for gas and dust, respectively, and $\zeta \approx R_\odot/2$. Since $h_p \ll h_g$ the second term is small and cannot compensate for the dropping off of the first at small $\tau(\zeta, z_1)$. From (15) it is seen that for $z_1 \ll h_g$ one has $\tau(\zeta, z_1) \approx \tau(\zeta, 0)$. The correction factor to (17) is therefore roughly

$$\xi \approx 1 - e^{-\tau(\zeta, 0)}. \tag{23'}$$

In Rayleigh scattering the ratio of the amount of light scattered by a single particle (atom or molecule) to the intensity of the incident light is given by (Allen, 1955)

$$\sigma_a = \frac{128\pi^5}{3\lambda^4}\left(\frac{n^2-1}{4\pi N}\right)^2, \tag{24}$$

where N is a number in cm^3 and n the index of refraction. For molecular hydrogen Born (Optics, 1937) gives the polarizability $(n^2-1)/4\pi N = 8.2 \cdot 10^{-25}$. The same value is obtained when one assumes, after Allen, that $n = 1.0001384$ for normal temperature and pressure and computes the corresponding value of $N = p/kT$. Thus

$$\varkappa = \frac{\alpha}{\rho} = \frac{\sigma_a N}{\rho} = 2.63 \cdot 10^{-5}\lambda^{-4}, \tag{24'}$$

where λ is now expressed in microns. From (15), for $\sigma = \text{const}$

$$\tau(\zeta, 0) = \frac{\varkappa\sigma}{2\beta} e^{\zeta/\beta R} E_1\left(\frac{\zeta}{\beta R}\right). \tag{25}$$

For a gas the parameter β is determined by its temperature, and for T independent of z it can be found from the barometric formula (3.5). Inserting $v=\sqrt{\frac{3\mathfrak{R}T}{\mu}}$, we obtain

$$\beta=h/R=\frac{\pi}{4V_c}\sqrt{\frac{3\mathfrak{R}T}{\mu}}. \tag{26}$$

The density of the gas should decrease with z as e^{-z^2/h^2}. But in view of the fact that the gas temperature may have been higher at large z than at the boundary of the dust layer, where it equals the temperature of the dust particles, one can assume as before that the density decreased approximately as $ae^{-z/h}$. Expression (26) with the value of T for the dust layer, and the data in Table 4 give us two relations between β and T and make it possible to determine both these quantities. They are given in Table 5.

TABLE 5

	Zone of Planet							
	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune
T, °K	145	100	84	67	35	18.4	9.3	6.1
β	0.019	0.021	0.023	0.024	0.033	0.033	0.033	0.033
$\tau(^1/_2R_\odot, 0)$.	0.85	1.04	1.15	1.3	1.5			
ξ_1	0.57	0.64	0.68	0.73	0.78			

The values of $\tau(R_\odot/2;\ 0)$ in the third row of the table were calculated from (25) for the surface gas density $\sigma=10^3$ to the distance of Jupiter and $10^3\ (R_0/R)^2$ for greater distances, in accordance with (21). This value of σ corresponds to a total cloud mass of $0.046M_\odot$, which is close to the value $0.05\,M_\odot$ adopted by us after examining the rate of growth of planetary giants (Chapter 12). The last row lists the values of the correction $\xi=\xi_1$ for $\lambda=1\mu$. As three-fourths of the energy of solar radiation belongs to the region of $\lambda<1\mu$, the correction $\xi^{1/4}$ for the cloud temperature is small. In the large-planet region it is insignificant, the maximum value (in the Jupiter zone) being -8%. It is slightly higher in the region of the Earth group. However, τ is computed only for Rayleigh scattering, without allowing for light absorption by various molecules. In reality τ should be larger and the temperature correction smaller than given by ξ.

The data cited in Table 4 were obtained on the assumption that $H\propto R$, i.e., $\beta=$ const. From Table 5 it is seen that this condition obtains in the large-planet region where σ falls off rapidly with R and $T\propto R^{-1}$. In the Earth-group region T decreases more slowly, approximately as $\propto R^{-0.6}$. Therefore $H\propto R^{1.2}$ and β increases with R. This departure from the condition $\beta=$ const ought to lead to temperatures higher than those indicated in Table 4. The correction is small and opposite in sign to the correction $\xi^{1/4}$. We will therefore limit ourselves to the uncorrected values of T given in Table 5.

14. Condensation of volatile substances on particles

Lebedinskii has demonstrated that solid particles can warm up thanks to the energy of random motion of massive protoplanetary bodies (1960). The bodies acquire relative velocities due to gravitational interaction among themselves. As they travel through the dust medium they undergo deceleration and impart to the dust particles an amount of energy capable of warming the latter by 5 — 30°K. Therefore hydrogen could not have condensed on the particles in the region of the large planets. As Table 5 indicates, even in the early phase of the cloud's evolution, before the formation of protoplanetary bodies, the temperature of the dust layer was fairly high and as far away as Neptune hydrogen condensation on the particles could not have taken place. Indeed, the condensation point of gaseous hydrogen is related to its density (saturation vapor density) as follows (Urey, 1958):

$$\lg \rho = -\frac{47.02}{T} - \lg T + 0.134 + 0.0363T. \qquad (27)$$

The actual gas density in the cloud's central plane is $\rho_0 = \sigma_g / H$, where H is determined from (26).

At the distance of Neptune and Jupiter, for the foregoing values of σ hydrogen condensation is possible only at temperatures below 4°K and slightly above 5°K, respectively. If we take $\sigma_g = 2400$ in the Jupiter zone, which corresponds to $\rho_0 = \rho^* = 10^{-9}$ g/cm^3, the condensation point of hydrogen rises only to 5.5°K. For the value $T = 35$°K obtained above for the Jupiter zone to drop to this level, the energy of solar radiation reaching this zone would have to decrease 1600 times. It has been conjectured that the outer parts of the cloud could have been screened due to thickening of the dust layer in the inner region, for instance as a result of turbulence or convection at the inner edge of the layer. However, in Chapter 2 we noted that the very high temperature gradient along R necessary for convection cannot have been achieved in this zone due to the Poynting-Robertson effect.

Certain perturbations could have appeared in the dust layer under the influence of the strongest corpuscular fluxes ejected by the active regions of the Sun. As yet it is not clear how efficiently these processes could have transported solid particles to large values of z. It is not even excluded that the fluxes flushed the dust particles out of the region.

Thus it seems that the mean radiant energy reaching the cloud was two to three orders of magnitude greater than the energy at which hydrogen could have frozen in the Jupiter zone. Therefore hydrogen could have entered into the composition of the solid particles only in the form of such compounds as CH_4, H_2O, and NH_3. In the large-planet region all the latter must have been in the solid state. It follows that the planets rich in free hydrogen, Jupiter and Saturn, must have acquired it mainly in the closing phase of growth when their mass had become large enough to hold the acquired hydrogen.

Chapter 5

GRAVITATIONAL INSTABILITY

15. Fundamental difficulties in the theory of gravitational instability in infinite systems

A medium is gravitationally unstable if newly developed density perturbations in it, however small, increase indefinitely with time due to gravity and disrupt the equilibrium.

Numerous works have been devoted of late to the problem of gravitational instability. The interest stems not only from the great cosmogonic significance of the problem, but also from the considerable mathematical and fundamental difficulties encountered in connection with instability in various systems. The linearized theory of instability, designed for a series of concrete cases, reduces to Jeans' well-known criterion (1929), which is in a certain sense evidence of its universality. On the other hand, it has been stressed in a number of works that its derivation is faulty, as the infinite homogeneous nonrotating medium considered by Jeans could not have been in equilibrium. In nonequilibrium (expanding or contracting) systems, small perturbations cannot lead to the formation of sufficiently dense condensations, such as galaxies (Lifshits, 1946; Bonnor, 1957).

In stellar and especially in planetary cosmogony, long periods of time present no difficulties. Here Newtonian analysis of bounded equilibrium systems is expedient. The simplest problem will then be to study instability in an infinite quiescent homogeneous medium. Jeans' criterion can be treated as a first approximation that gives us, in the simplest cases, the correct order of the critical wavelength of the perturbation responsible for instability. Since among the forces counteracting instability allowance is made only for gas pressure in the perturbing wave, Jeans' criterion gives us the lower limit of the critical wavelength.

The main difficulty with Jeans' theory is due to a gravitational paradox: for an infinite homogeneous medium there is no gravitational potential. From Poisson's equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=-4\pi G\rho \qquad (1)$$

for $\rho\neq 0$, it follows that both the potential Φ and the gravitational attraction increase indefinitely with distance. This difficulty is circumvented in Jeans' theory as well in its subsequent extensions by applying Poisson's equation not to the entire medium but only to the perturbations, to the departures of the density $\delta\rho$ from its mean value ρ. It is assumed that in a "truly" infinite, homogeneous, quiescent system, there should be no

gravitational attraction, as it lacks pressure gradient and accelerations. Otherwise it would not be at rest.

Such an infinite system cannot be obtained by a limiting procedure from a finite system (such as a spherical one) for $R \to \infty$. Such a statement of the problem cannot be applied to gravitationally bound finite systems, for which it is necessary that Poisson's equation be satisfied in the Newtonian approximation and its analog be satisfied in the relativistic approximation.

The simplest and clearest derivation of Jeans' criterion can be obtained by considering the forces acting upon an element of the medium. Two forces arise in the propagation of a perturbation wave: gravitational attraction, related to the density perturbation $\delta\rho$; and the gas pressure force, related to the density gradient. For a plane wave at a point with displacement ξ, the former is given by (per unit mass)

$$\delta F_g = 4\pi G \rho_0 \xi, \tag{2}$$

and the latter by

$$\delta F_p = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \rho}{\partial x} = -\frac{c^2}{\rho}\frac{\partial \delta\rho}{\partial x} \approx +c^2\frac{\partial^2 \xi}{\partial x^2}, \tag{2'}$$

since

$$\rho = \rho_0 + \delta\rho, \qquad \delta\rho = -\rho\frac{\partial \xi}{\partial x},$$

and the displacement ξ is assumed to be small. The velocity of sound is denoted by c. For a sinusoidal perturbation

$$\xi = \xi_0 \sin\left(\omega t + \frac{2\pi x}{\lambda}\right),$$

$$\frac{\partial^2 \xi}{\partial x^2} = -\frac{4\pi^2}{\lambda^2}\xi. \tag{3}$$

The instability condition

$$\delta F_g > -\delta F_p \tag{4}$$

leads to Jeans' well-known criterion for the critical wavelength of perturbations:

$$\lambda_c^2 = \frac{\pi c^2}{G\rho_0}. \tag{5}$$

Instability will develop for any perturbation of wavelength $\lambda > \lambda_c$.

Further progress in the linearized theory of gravitational instability was associated mainly with attempts to allow for rotation and the magnetic field. Chandrasekhar (1955) considered the uniform rotation of an infinite homogeneous system. Bel and Schatzman (1958) obtained a similar result for a system of homogeneous density but in nonuniform rotation. They analyzed perturbations propagating in a plane perpendicular to the axis of rotation z, symmetric with reference to this axis and independent of z (cylindrical). The instability condition they obtained has the form

$$4\pi G\rho > \frac{2\omega}{R}\frac{d}{dR}(\omega R^2) + \frac{4\pi^2 c^2}{\lambda^2} + \frac{c^2}{4R^2}. \tag{6}$$

In these works as in many others dealing with rotating systems, Poisson's equation is applied only to density perturbations. It is assumed that the unperturbed medium is in equilibrium. But the question of how equilibrium is established is generally disregarded. In contrast with quiescent, infinite, homogeneous medium, in rotating systems a centrifugal force is present.

One can suppose that this force is balanced by the attraction of the matter contained in a cylinder of radius R which is infinite along the z axis. This means that we are applying Poisson's equation to the homogeneous medium along R and at the same time may not apply it along the z axis, for the same reasons as in Jeans' theory — because there are no forces capable of counteracting gravity in this direction. The condition of equilibrium in the R direction establishes the relation between ρ and ω. For $\rho = \text{const}$, $\omega^2 = 2\pi G\rho$. Inserting this value of ω in (6), we find that the critical density necessary for instability must be at least twice the actual density. Consequently, in this case gravitational instability will not arise when perturbations propagate in a plane perpendicular to the z-axis.

The instability condition (6) presupposes $\varrho = \text{const}$, $\omega \neq \text{const}$. Such a system cannot be in equilibrium. To achieve equilibrium additional masses of nongaseous nature (stars), with a density ρ_s dependent only on R, must be introduced in the system (Simon, 1962 a). Even so, the instability condition (6) will not be satisfied.

Thus for the systems under consideration the condition of equilibrium (based, naturally, on the use of Poisson's equation) seems to be incompatible with the condition of gravitational instability. A similar result was obtained by Jeans for a finite spherical mass in equilibrium — it cannot break down by gravitational instability into separate components. The theory of gravitational instability in a rotating medium of infinite extension along z is chiefly of mathematical interest, as there are no real systems to which it could be applied. Nonetheless it represents an important step toward understanding gravitational instability in real finite systems.

16. Gravitational instability in flat systems with nonuniform rotation

Real astronomical systems of finite dimensions fall into two main categories — spherical and flat. We saw that a spherical equilibrium system of any finite radius cannot break down into separate clusters since the critical wavelength is close to the diameter of the system. Instability in expanding and contracting systems is examined in many works; we will not be concerned with it, since the protoplanetary cloud belonged to the category of flat rotating systems. In flat rotating systems in equilibrium, by contrast with spherical ones, gravitational instability will arise if the density of matter in the system exceeds the critical value. Due to the complexity of the problem, however, for these systems no one has constructed even a linearized theory of propagation of small perturbations.

Uniform rotation has been investigated by Fricke (1954), but he was unable to avoid arbitrary assumptions. We have demonstrated that Bel and Schatzman's attempt to apply condition (6) to flat systems is untenable: too low a value is obtained for the critical density. It was found that it is

possible to effect a transition from two-dimensional cylindrical rotating systems to flat ones by multiplying the term $4\pi G\rho$ in the left-hand side of (6) by the function $f(\lambda/H) < 1$; the latter function was calculated (Safronov, 1960c).

It will be seen that condition (6), like the instability condition (4), represents the balance of the forces acting upon an element that has been shifted radially by the perturbing wave through a distance $\delta R = 1$ without change of angular momentum with respect to the center of the system. Since the maintenance of equilibrium in a homogeneous medium with nonuniform rotation requires us to assume the presence in the system of additional masses of a different nature (such as stars), it is simpler to consider the variation of F_g and F_c at a certain fixed point in space (at the given R) than to track the displaced element.

From Poisson's equation it is easy to find the attraction per unit mass of an infinite cylinder:

$$F_g = -2Gm/R, \tag{7}$$

m being the mass enclosed in a cylindrical layer of radius R and height 1 cm. For radial perturbation the mass m will change by $\delta m = -2\pi R\rho\delta R$. Consequently

$$\delta F_g = -2G\delta m/R = 4\pi G\rho\delta R. \tag{8}$$

This expression is the left-hand side of (6). The centrifugal force is given by $F_c = \omega^2 R = k^2/R^3$, where $k = \omega R^2$ is the angular momentum density. The element moves over the distance R with the angular momentum it possessed in unperturbed displacement at the distance $R - \delta R$. Therefore

$$\delta F_c = -\frac{1}{R^3}\frac{dk^2\delta R}{dR} = -\frac{2\omega}{R}\frac{d}{dR}(\omega R^2)\,\delta R. \tag{9}$$

Since in real systems $\frac{d(\omega R^2)}{dR} > 0$, the force δF_c is negative and tends to return the displaced element to its previous position. As we know, rotation stabilizes the system. Expression (9) represents the first term in the right-hand side of condition (6) for $\delta R = 1$. The second term of (6) is the gas pressure gradient due to perturbation, and it too represents a force acting opposite to the displacement. The last term of (6) can be disregarded, since only perturbations with $\lambda < R$, for which this term is much smaller than the preceding one, are of practical interest.

When one passes from a system infinite along the z-direction to flat systems, only the term related to gravity in the left side of (6) changes. It is no longer worthwhile to use Poisson's equation to determine the component of gravitational attraction along R of a ring of density $\delta\rho$, as it now includes the additional term $\partial^2\delta\varphi/\partial z^2$. It would therefore be more expedient to calculate δF_g directly. The expression for δF_g is somewhat cumbersome, as it contains elliptic integrals which, moreover, lie under the integral sign (Safronov, 1960c). However, in the case of a ring the first-order term which interests us in δF_g is equal, for $\lambda \ll R$, to the value of δF_g for an infinite cylinder perpendicular to R and z and having a cross section equal to that of the ring containing the point R_0. We will confine ourselves to the simpler evaluation of δF_g for the cylinder, which corresponds to the case of a plane wavefront. Consider a sinusoidal wave perturbation having amplitude δR at the point R_0:

$$\xi = \delta R \cos\frac{2\pi r}{\lambda}, \qquad \delta\rho = -\rho\frac{\partial\xi}{\partial r} = \frac{2\pi\delta R}{\lambda}\sin\frac{2\pi r}{\lambda}\rho, \tag{10}$$

where $r = R - R_0$. From (7), the attraction exerted at the point R_0 by an infinite elementary cylinder of cross section $drdz$ and density $\delta\rho$ whose generator is perpendicular to the R and z axes is given by $2G\delta\rho drdz/\sqrt{r^2+z^2}$; its component along R is given by

$$d\delta F_g = \frac{2G\delta\rho r dr dz}{r^2+z^2} = \frac{4\pi G\rho\delta R}{\lambda}\sin\frac{2\pi r}{\lambda}\frac{r dr dz}{r^2+z^2}. \tag{11}$$

For ρ we take its value averaged over z within the homogeneous thickness $2h$. The limits of integration over r will be $+\lambda/4$, the maximum distance reached by the perturbation, which has a first maximum ξ at R_0, and correspondingly $-\lambda/4$.* Then

$$\delta F_g = \frac{4\pi G\rho\delta R}{\lambda}\int_{-\lambda/4}^{+\lambda/4}\int_{-h}^{+h}\sin\frac{2\pi r}{\lambda}\frac{r dr dz}{r^2+z^2} = 16\pi G\rho\delta R\int_0^{\lambda/4}\sin\frac{2\pi r}{\lambda}\operatorname{arc\,tg}\frac{h}{r}dr. \tag{12}$$

Setting

$$\frac{4r}{\lambda} = x, \qquad \frac{\lambda}{2h} = \frac{\lambda}{H} = \zeta, \tag{13}$$

we find that

$$\delta F_g = 4\pi G\rho f(\zeta)\delta R, \tag{14}$$

where

$$f(\zeta) = \int_0^1 \sin\frac{\pi x}{2}\operatorname{arc\,ctg}\frac{\zeta x}{2}dx. \tag{15}$$

Thus the gravitational instability condition for a flat system in nonuniform rotation can be written as

$$4\pi G\rho f(\zeta) > \frac{2\omega}{R}(\omega R^2)' + \frac{4\pi^2 c^2}{\lambda^2}. \tag{16}$$

The attraction of the central body (Sun) is present in the form of the function $\omega(R)$. The function $f(\zeta)$ has the following values:

ζ	0.2	2	4	6	8	10	14	20
$f(\zeta)$ . . .	0.96	0.64	0.43	0.34	0.28	0.23	0.172	0.124

Hence the correction to the critical density is significant and depends on the ratio of the wavelength of the perturbation to the thickness of the layer.

Let us find the value of ζ for which the critical density necessary for gravitational instability is minimum. From (3.30)

* The lower limit of integration r_1 will depend on the nature of the perturbation (single wave or train), and is not completely defined. But the result is not strongly affected by this: for $r_1 = -3\lambda/4$ the value of δF_g will be 10% higher than obtained above, and for $r_1 = -\infty$ not more than 18% higher. It is interesting to note that in flat systems, unlike systems infinite along z, the perturbation $\delta\rho$ begins to excite the gravitational force δF_g at the point R_0 not at the instant when it reaches R_0 but rather when the perturbation appears at any distance, however large, from R_0. But the maximum perturbation occurs for maximum displacement δR to R_0.

$$H = \frac{\sigma}{\rho_0} = \sqrt{\frac{2\mathfrak{R}T}{\pi G \mu \rho_0}} \cdot \mathfrak{I}, \tag{17}$$

where

$$\mathfrak{I} = \frac{1}{2} \int_0^1 \frac{d\eta}{\sqrt{1 - \eta - \frac{1}{3}\eta^* \ln \eta}}, \qquad \eta^* = \frac{\rho^*}{\rho_0}, \qquad \rho^* = \frac{3M}{4\pi R^3}. \tag{18}$$

Evaluation of the integral gives us the following dependence of $\mathfrak{I}$ on ϱ/ϱ^*:

ρ/ρ^*	1/3	4/3	10/3	5	10
$\mathfrak{I}$	0.66	0.87	0.94	0.96	0.975

Substituting $H = \lambda/\zeta$ and $\mathfrak{R}T/\mu = c^2/\gamma$ in (17), we obtain

$$\frac{4\pi^2 c^2}{\lambda^2} = 4\pi G \frac{\gamma \pi^2 \rho_0}{2\mathfrak{I}^2 \zeta^2}, \tag{19}$$

where

$$\gamma = c_p/c_v.$$

For a system whose rotation is determined mainly by the attraction of the central mass (solar system, outer parts of the Milky Way),

$$\omega R^2 = \sqrt{GMR}, \qquad \frac{2\omega}{R}(\omega R^2)' = \omega^2 = \frac{4}{3}\pi G \rho^*. \tag{20}$$

Then the stability condition (12) can be written as

$$\rho > f^{-1}(\zeta)\left(\frac{\rho^*}{3} + \frac{\pi^2 \gamma}{2\mathfrak{I}^2}\frac{\rho_0}{\zeta^2}\right). \tag{21}$$

The quantity ϱ represents the density of a homogeneous layer of thickness $2h$. A real rotating cloud with exponential density distribution $\varrho(z)$ will have a lower concentration toward the $z = 0$ plane. It will produce the same δF_g along R as a homogeneous layer would for $\varrho_0 > \varrho$.

Calculations show that $\varrho \approx 0.9\,\varrho_0$ and is only weakly dependent on ζ. Recalling this and using the numerical values listed above for $f(\zeta)$ and $\mathfrak{I}$, one can find the critical value ϱ_0 satisfying the instability condition (21). The results of calculations for $\gamma = 1$ are cited below and in Figure 2.

ζ	4	6	8	10	15
$\rho_{0_{CT}}/\rho^*$ . . .	6.8	2.3	2.1	2.2	2.4

Thus the critical density required for gravitational instability, which depends, as we know, on λ, is minimum when the wavelength of the perturbation is eight times the cloud thickness H. As λ decreases the critical density increases due to the increase of the second term in (21), which is related to the usual Jeans criterion. As λ increases the main factor in (21) becomes the first term, which is related to the rotation of the system.

Here the critical density increases due to the increase of the function $f^{-1}(\zeta)$, which shows how many times smaller the attraction of a flat ring is than the attraction of a tube constructed on this ring which is infinite along the z-direction. The minimum critical density $\varrho_{cr} = 2.1\,\varrho^*$ is more than 6 times greater than the critical density $\varrho^*/3$ obtained by Bel and Schatzman for the two-dimensional case. It is also larger than the value obtained by Chandrasekhar for uniform rotation.

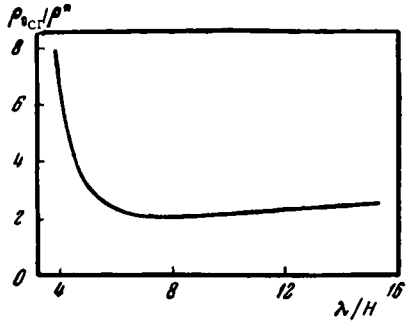

FIGURE 2. Dependence of the critical density for gravitational instability in a flat layer rotating around a central mass M on the wavelength of radial (ring) perturbation:

H — layer thickness; $\rho^* = \frac{3M}{4R^3}$.

Consider the influence of the magnetic field and viscosity on the instability condition. If the perturbation is propagating perpendicularly to the magnetic field (in our case this corresponds to a toroidal field), then instead of c^2 in the instability condition (16) we have the sum $c^2+v_a^2$, where $v_a = H/\sqrt{4\pi\rho}$ is the Alfvén velocity.

In the linearized theory the introduction of a nonzero viscosity for the medium leads to the exclusion of the term related to the system's rotation from (16), while the introduction of magnetic viscosity leads to the exclusion of the factor v_a^2 in the last term (Pacholczyk and Stodolkiewich, 1960). Allowance for the thermal conductivity causes the velocity of sound to change from adiabatic to isothermal (Kato and Kumar, 1960). The coefficients of viscosity (ordinary and magnetic) and of thermal conductivity do not enter into the instability condition. However small they are (provided they are nonzero), the corresponding terms will not appear. Since under real conditions the viscosity, although very low, is not rigorously zero and the electrical conductivity is not infinite, it is sometimes formally inferred that neither the system's rotation nor the magnetic field affect the stability of the medium, and that the ordinary Jeans criterion (or Ledoux criterion for a flat system) is to be used. This result is strictly due to the fact that the perturbations are assumed to be infinitesimal. They develop over an indefinite period, and thus the viscosity of the medium and the thermal conductivity would eventuaally produce the indicated effect. Here, however, we encounter a difficulty which is not accorded the attention it deserves. In general, a viscous medium with differential rotation is not in equilibrium. Hence to state the problem of the instability of such a medium with respect to infinitesimal perturbations assuming that the unperturbed medium would be in equilibrium is wrong in itself. Moreover under real conditions perturbations are always finite, and in many astronomical systems the viscosity is usually so low that it is not a factor. The instability criterion for such systems should have the form (16) with the addition of v_a^2, i. e., both rotation and the magnetic field are important in this relation.

A similar situation obtains in regard to the influence of the magnetic field for an infinite electric conductivity, when the field is not rigorously perpendicular to the perturbation. Formally the quantity v_a drops out of the instability condition even for a very small component H_z of the field along the direction of propagation of the wave. Yet in reality for finite perturbations a field nearly perpendicular to the perturbation will set up a resistance to it, contributing to the stability of the medium.

Thus for finite perturbations the gravitational instability conditions differ in a number of essential respects from the criteria obtained assuming infinitesimal perturbations. This fact must be taken into account in cosmogonic applications of the theory of gravitational instability.

17. Growth of perturbations with time

It is usually understood that for $\lambda > \lambda_c$ the perturbation will lead to unlimited compression of the flat layer. In particular, this conclusion is drawn by Simon (1962b). Looking at the equation of motion, Simon found that for a sufficiently large time t the density increase at the center of the layer is approximated by the function $t^{-3/2}e^{kt}$. However, this conclusion is wrong. The equation studied by Simon applies only to small perturbations and is unsuitable for large periods of time in which the density becomes significantly higher than its initial value. For displacements of the same order as λ, expression (2) for the gas pressure at a point with initial coordinate x and displacement $\xi(x)$ has the form (Safronov, 1964a)

$$\delta F_p = -\frac{1}{\rho}\frac{dp}{d\rho}\frac{d\rho}{\partial(x+\xi)} = -\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\frac{(dp/d\rho)_0}{\rho\left(1+\frac{\partial\xi}{\partial x}\right)}\frac{\partial\rho}{\partial x}, \tag{22}$$

where $\gamma = c_p/c_v$.

The continuity equation gives the relation

$$\rho\left(1+\frac{\partial\xi}{\partial x}\right) = \rho_0. \tag{23}$$

Introducing ρ into (5) as in the above expression and setting $(dp/d\rho)_0 = c^2$, we obtain

$$\delta F_p = c^2\left(1+\frac{\partial\xi}{\partial x}\right)^{-1-\gamma}\frac{\partial^2\xi}{\partial x^2} = c^2\left(\frac{\rho}{\rho_0}\right)^{1+\gamma}\frac{\partial^2\xi}{\partial x^2}. \tag{24}$$

The equation of motion will therefore be as follows:

$$\frac{\partial^2\xi}{\partial t^2} = 4\pi G\rho_0\xi + c^2\left(1+\frac{\partial\xi}{\partial x}\right)^{-1-\gamma}\frac{\partial^2\xi}{\partial x^2}. \tag{25}$$

Simon's equation lacks the term $\partial\xi/\partial x$, or, which amounts to the same, the factor $(\rho/\rho_0)^{1+\gamma}$ for $\partial^2\xi/\partial x^2$. As long as the displacements are small, $\rho \approx \rho_0$; since $\frac{\partial^2\xi}{\partial x^2} \sim -\frac{\xi}{(\lambda/2\pi)^2}$, for $\lambda > \lambda_c$ the right-hand side of (25) is positive and increases with ξ. The perturbation is constantly increasing in strength. But when ρ becomes distinctly larger than ρ_0, the second (negative) term in the right-hand side increases more rapidly than the first and contraction ceases. In the case of sinusoidal perturbation and $\gamma = 1$, by the time $\rho = \rho_0\lambda/\lambda_c$ the acceleration $\partial^2\xi/\partial t^2 = 0$ and the rate of contraction begins to fall off. Simon's inference that instability develops even for $\lambda < \lambda_c$ is based on a misunderstanding.

A flat layer cannot contract indefinitely, even if all the heat generated is emitted in the process (isothermal contraction). According to Ledoux (1951), an infinite flat isothermal layer of specified surface density σ has the following thickness when in equilibrium:

$$H = \frac{2c^2}{\gamma\pi G\sigma}. \tag{26}$$

In reality H does not depend on γ ($c^2 \propto \gamma$ in the numerator) and is determined by the temperature of the layer. The above expression refers to an isolated layer. But a layer formed by gravitational instability will be surrounded by an infinite attracting medium which stretches the layer, increasing H and decreasing the density ϱ_c in its central plane. Recalling that

$$\sigma = \rho_c H = \rho_0 \lambda,$$

we find from (26), in view of (4), that

$$\frac{\rho_c}{\rho_0} \leqslant \frac{\gamma \pi^2}{2} \frac{\lambda^2}{\lambda_c^2}. \tag{27}$$

For $\lambda \gg \lambda_c$ one can take the equality sign above, since $H \ll \lambda$ and the attraction of the surrounding medium is small.

An isothermal flat layer in equilibrium is not bounded. Consequently, for a perturbed region with initial dimensions λ the contraction stage should give way to a stage in which its outer portions expand and fuse with the surrounding medium. According to the linearized theory of instability, for $\lambda > \lambda_c$ the rate of wave propagation becomes imaginary, and no perturbation will propagate beyond the area of developing instability. However, the picture changes radically when the process deviates from linearity. No closing up of the localization of the perturbed region takes place in the case of a flat layer. The expansion wave penetrates to the surrounding medium, where it produces a perturbation which continues to travel on.

The foregoing considerations do not rob Jeans' theory of gravitational instability of all cosmogonic significance, but they do significantly alter our conception of the nature of the development of instability. A single wave perturbation is not sufficient for the density to increase indefinitely. It merely leads to the formation of a flat layer. But if a new perturbation of wavelength $\lambda' > \lambda'_c$ were now to travel along this layer, it would again give rise to gravitational instability, leading to the formation of a contracting cylindrical region (quantities referring to the flat layer will be designated by a prime). According to Ledoux for $\gamma = 1$

$$\lambda_c'^2 = \frac{2\pi c^2}{G\rho_c}. \tag{28}$$

Earlier we found that when a perturbation travels along the layer the gravitational attraction $\delta F'_g$ is given by expression (14):

$$\delta F'_g = 4\pi G \rho_c f(\zeta) \xi. \tag{29}$$

The development of perturbations inside the layer proceeds as in a medium infinite in all directions, except that when describing them the gravitational constant G should be replaced by $Gf(\zeta)$. The presence of the extra factor of two in the numerator in Ledoux's criterion (28), compared with Jeans' criterion (5), is due to the fact that for $\gamma = 1$, from (26) and (28), $\lambda'_c/H = \pi$ while $f(\pi) = 0.5$.

If the thickness H remained constant throughout the process of development of the perturbation with $\lambda' > \lambda'_c$ inside the layer, then, as in the preceding instance, the perturbation could not have caused unlimited

contraction. For maximum density growth we would have an expression similar to (27) with an extra factor $f(H'/H)/f(\lambda'/H)$ on the right, of the order of two. The width of the contracting zone would have reached the value $H' < H$. In reality contraction proceeds in both directions and the configuration tends to an infinite circular cylinder.

The cylinder represents an intermediate case between a flat layer and a sphere. Earlier it was shown that for any γ a flat layer is stable. On the other hand it is well known that for $\gamma < 4/3$ a sphere will be unstable and will contract indefinitely. Applying similar arguments to the cylinder, one finds that the critical value of the adiabatic index $\gamma = 1$. Indeed, for a cylinder of radius r

$$F_g = -\frac{2Gm(r)}{r},$$

$$F_p = -\frac{1}{\rho}\frac{\partial p}{\partial r} \sim \frac{1}{\rho}\frac{p}{r} \propto \frac{\rho^{\gamma-1}}{r}.$$

If the cylinder is subjected to radial contraction, then $m(r) = \text{const}$, $\rho \propto r^{-2}$, and

$$F_g \propto r^{-1}, \qquad F_p \propto r^{-2(\gamma-1)-1}. \tag{30}$$

For instability it is necessary that F_g change faster than F_p, i.e., that

$$2(\gamma-1)+1<1 \quad \text{and} \quad \gamma<1. \tag{31}$$

For all $\gamma > 1$ the cylinder will be stable under radial contraction. The case $\gamma = 1$ corresponds to neutral equilibrium: if there was equilibrium prior to contraction, contraction will not disrupt it. It is therefore desirable to determine what equilibrium configurations obtain for an isothermal cylinder.

For an infinite cylinder the condition of equilibrium in the radial direction

$$dp = -g\rho dr, \quad \text{where } g = \frac{2Gm(r)}{r}, \qquad m(r) = 2\pi\int_0^r r\rho dr \tag{32}$$

leads to a second-order differential equation (Safronov, 1964a; Ozernoi, 1964):

$$\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} - \frac{1}{\rho}\left(\frac{d\rho}{dr}\right)^2 + \frac{2\pi G\rho^2}{c^2} = 0. \tag{33}$$

This equation can be reduced to the Euler equation. For the boundary conditions

$$r=0, \qquad \rho=\rho(0), \qquad \frac{d\rho}{dr}=0. \tag{34}$$

Its solution is given by

$$\rho = \rho(0)(1+a^2r^2)^{-2}, \tag{35}$$

where

$$a^2 = \frac{\pi G\rho(0)}{2c^2}. \tag{36}$$

The mass of a unit cross section within radius r is given by

$$m(r)=\frac{\pi r^2}{1+a^2r^2}\rho(0). \tag{37}$$

The total mass of a unit cross section of the equilibrium cylinder,

$$m=m_c=2c^2/G \tag{38}$$

is independent of $\varrho(0)$ and has a unique value.

Hence the isothermal cylinder is not an equilibrium configuration: it contracts for $m > m_c$ and expands for $m < m_c$. Equilibrium is possible only for m exactly equal to m_c and, as we saw earlier and as is also evident from (35), it is neutral with respect to radial contraction. In the latter solution $\varrho(0)$ can be varied at will. Correspondingly a varies but $m=m_c$ continues to hold. Expression (38) for m_c can be obtained more simply with the aid of the virial theorem up to a factor of the order of unity (McCrea, 1957).

Let us return to the flat layer. For the critical wavelength the mass of a unit cross section of a contracting band, from (26) and (28), is given by

$$m=\lambda'_c\rho_c H=\lambda_c'^2\rho_c\frac{H}{\lambda'_c}=\frac{2c^2}{G}=m_c. \tag{39}$$

Consequently the gravitational instability condition in a flat layer ($\lambda'>\lambda'_c$) leads to $m>m_c$, i.e., it is also the condition for unlimited contraction stemming from the instability of the cylinder. We note that the condition for instability for a cylinder with respect to perturbations along its axis, obtained by Dibai (1957), is also fulfilled:

$$m'_c=\frac{\mu_1^2c^2}{4G}, \tag{40}$$

where $\mu_1=2.4$. The cylinder breaks up into separate clusters which contract even more rapidly.

A similar picture will obtain in the case of an annular perturbation inside a flat rotating layer. The supplementary term in (16) related to rotation will be a significant factor only at the early phase of development of the perturbation when it is of the same order as the next term characterizing the gas pressure. But when the width $2r_c$ of the ring shrinks by more than half, the increase in F_c (in absolute magnitude) will be distinctly slower than the increase in F_p, as $F_c\propto\xi<\lambda/4$ while $F_p\propto r_c^{-1}$ (from (30)). Since condition (16) is fulfilled at the start, it should certainly be fulfilled later, ensuring the further growth of the perturbation.

At the initial phase, when the perturbation is still small, it increases exponentially as $\xi=\xi_0e^{\omega t}$,

$$\omega^2=4\pi G(\rho_{\text{in}}-\rho_{\text{cr}})f(\zeta)=3\omega_c^2(\rho_{\text{in}}/\rho^*-\rho_{\text{cr}}/\rho^*)f(\zeta). \tag{41}$$

where ω_c is the angular velocity of revolution around the Sun. The perturbation will develop very rapidly even if ρ_{in} is only slightly greater than ρ_{cr}. Let $\rho_{\text{cr}}=2.1\rho^*$, $\rho_{\text{in}}=2.2\rho^*$, and $f(\zeta)=0.28$. Then $\omega=0.29\omega_c$ and at the Earth's distance from the Sun the time required for e-fold growth of the perturbation

is given by $\omega^{-1} = 0.55P$, where $P = 2\pi/\omega_e$ is the period of revolution around the Sun. Within 10 revolutions the perturbation ξ_0 will increase 10^8 times. Twenty years would be sufficient for even a very small perturbation ($\xi_0 = 1$ cm), set up by a corpuscular stream possessing energy characteristic of large solar flares, to grow to a considerable size ($\xi = H \approx 10^8$ cm) at the Earth's distance from the Sun.

Thus cosmogonically the gravitational instability in the flat layer revolving around the Sun developed within a very short time, of the order of a few tens of periods of revolution.

Chapter 6

FORMATION AND EVOLUTION OF PROTOPLANETARY DUST CONDENSATIONS

18. Mass and size of condensations formed in the dust layer

The inference that the dust layer revolving around the Sun disintegrated into a large number of dust condensations was first stated, independently and almost simultaneously, by Edgeworth (1949) and by Gurevich and Lebedinskii (1950). Basing himself on Maxwell's well-known research into the stability of Saturn's rings (1890), Edgeworth conjectured that for a density of $0.04\varrho^*$ the layer becomes unstable, with random eddies developing inside it; these eddies give rise to density fluctuations which grow and transform for $\varrho > 3\varrho^*$ into roughly spherical nondisintegrating condensations. However, the probability for such large random fluctuations is extremely low. We saw earlier that gravitational instability develops inside the layer at significantly higher densities: $\varrho > 2.1\varrho^*$. Maxwell's result is rigorously valid only for material points situated on a ring and is not applicable to a large number of particles colliding with each other and forming a virtually dense medium. The Maxwellian ring breaks down when the amplitude of oscillation of the material points equals the mean distance between adjacent points and collision between them becomes possible. But in the layer, collisions between real particles have no effect whatsoever on its stability.

Gurevich and Lebedinskii estimated the densities and sizes of condensations after analyzing the energy aspects. They determined the size and density that a spheroidal region formed in an unperturbed disk must have if it is to hold together by internal gravitational attraction when the material of the disk surrounding it is removed. The fundamental condition was obtained from the virial theorem, both the random relative velocities of the particles and the ordered velocities associated with the cloud's differential revolution around the Sun being taken into account. It was found that the density of this spheroidal region (condensation) must be one order greater than the "spread out" density ϱ^* of the Sun, and that sizes in the plane of the layer should be 13 times greater than sizes at right angles to it, i.e., than the uniform thickness H of the layer. The first result agrees with estimates of the Roche density ρ_R. The second was not known previously.

This method gives the following values for the mass m_0 and semimajor axis a_0 of the condensation:

$$m_0 \approx \frac{\sigma^3}{2\rho^{*2}}, \qquad a_0 \approx \frac{\sigma}{2\rho^*}, \tag{1}$$

where σ is the surface density of the dust matter in the layer.

A certain inaccuracy is introduced when the condensation is assumed to be spheroidal. The actual condensation could not have formed from a spheroidal region, since different directions inside the central plane of the layer were not equivalent. As a density estimate applies to an isolated condensation, in the presence of surrounding medium it will give an excessively high value for the density of the condensation compared with that of the background. It is evident that such large density fluctuations could not have arisen in the absence of gravitational instability. Of all the possible perturbations it is best to consider radial annular perturbations, as they are the only ones which do not disintegrate upon differential rotation and which can increase in intensity during many periods of revolution around the Sun.

In Chapter 5, Section 16, we saw that for a density $\varrho > \varrho_{cr}$ radial perturbation will lead to the formation of a contracting ring. In the contraction of the ring the angular momentum of its material with reference to the Sun is conserved, and therefore its orbital velocity is changed. The outer half moves toward the Sun and accelerates, while the inner half moves away and slows down. When the width of the ring shrinks by one half, the linear velocity of rotation of all its parts becomes uniform, and by the time it shrinks to one quarter the ring revolves as a rigid body. When the width of the ring is much smaller than its radius, the condition that it disintegrates into separate condensations is close to the condition of disintegration of an infinite cylinder. The condition for gravitational instability of an infinite homogeneous fluid cylinder of radius R_0 with respect to longitudinal perturbations was obtained by Dibai (1957) in the form

$$\lambda > \lambda_0 = \frac{2\pi R_0}{\mu_1}\left[\frac{4\pi G\rho R_0^2}{\mu_1^2 c^2} - 1\right]^{-1/2}. \tag{2}$$

Taking the mass of a unit cross section of the cylinder to be equal to the mass of unit cross section of a ring with initial width ζH,

$$\pi\rho R_0^2 = \zeta\sigma H$$

and, from (5.17),

$$c^2 = \frac{\gamma RT}{\mu} = \frac{\gamma\pi G\sigma H}{2\mathfrak{J}^2},$$

we obtain

$$\lambda_0 = \frac{2\pi}{\mu_1} R_0 \left(\frac{8\zeta\mathfrak{J}^2}{\pi\gamma\mu_1^2} - 1\right)^{1/2}. \tag{3}$$

For $\zeta = 8$, $\mu_1 = 2.4$, $\gamma = {}^5/_3$ and $\mathfrak{J} = 0.92$, we obtain $\lambda_0 = 3R_0$.

The minimum mass of the condensation is given by

$$m = \zeta\sigma H\lambda_0 = 3\zeta\sigma H R_0. \tag{4}$$

For R_0 we can take the geometric average between the half-width $\zeta H/2n_1$ of the ring and its half-thickness ($\sim H/2$) at the instant of disintegration. Quantity n_1 shows how many times the width of the ring has decreased at the instant of disintegration. Then

$$m \approx \frac{3\zeta^{3/2}}{2\sqrt{n_1}} \cdot \frac{\sigma^3}{\rho_0^2} = \frac{3\zeta^{3/2}}{\sqrt{n_1}} \left(\frac{\rho^*}{\rho_0}\right)^2 m_0. \tag{4'}$$

For $\rho_0 = 2.5\rho^*$, $\zeta = 8$ and $n_1 = 4$ the minimum mass of a condensation turns out to be three times greater than the mass obtained according to the method of Gurevich and Lebedinskii. The initial radius of the condensation in the radial direction is given by

$$a = \frac{\zeta H}{2n_1} = \frac{\zeta\rho^*}{n_1\rho_0} a_0. \tag{5}$$

For the same values of ρ_0 and n_1 we obtain $a = 0.8a_0$. The minimum size of a condensation in the direction of orbital motion will be 1.5 times greater than radially. The actual size may be slightly larger. Attention should be drawn to the close agreement between estimates for mass and radius as obtained by the different methods. Note, too, that gravitational instability inside the dust layer and dissolution of the condensations could have taken place for different values of the parameters ξ, n_1 and ρ_0 in (4'). Thus there could have been considerable divergences in the initial masses of the condensations.

Henceforth we will use the following mean initial values:

$$m_0 = \frac{\sigma^3}{\rho^{*2}}, \qquad a_0 = \frac{\sigma}{2\rho^*}. \tag{6}$$

For the terrestrial zone this yields $m_0 = 5 \cdot 10^{16}$ g and $a_0 = 4 \cdot 10^7$ cm for $\sigma = 10$ g/cm^2; at the distance of Jupiter the values are respectively 10^{22} g and 10^{10} cm for $\sigma = 20$ g/cm^2.

The internal gravitational forces of the evolving condensation exceed the external forces. It therefore begins to contract until its gravity is balanced by internal pressure and by the centrifugal force, which increases with contraction. The decrease in the relative velocities of the particles caused by inelastic collisions is accompanied by contraction of the condensation along the axis of rotation. However, frequent external perturbations resulting from encounters and collisions among condensations tend to prevent unlimited slowing down of the particles and unlimited flattening of the condensation.

The equatorial radius r of the condensation and its angular velocity of rotation ω before contraction (subscript 0) and after contraction (subscript 1) are related by the condition of angular momentum conservation and the equilibrium condition:

$$r_0^2\omega_0 = r_1^2\omega_1, \qquad r_1^3\omega_1^2 = \xi Gm, \tag{7}$$

where m is the mass of the condensation and ξ is a coefficient which depends on the form of the condensation. Hence

$$r_1 = \frac{r_0^4\omega_0^2}{\xi Gm}. \tag{8}$$

For a homogeneous spheroidal condensation revolving as a rigid body (Maclaurin spheroid), $\xi \approx 1/2$ for the axial ratio $c/a = 0.6$, $\xi \approx 1$ for $c/a = 1/3$, and $\xi \approx 2$ for $c/a = 0.1$. In the other extreme case of a Roche model (nearly all the mass concentrated at the center), $\xi = 1$. Therefore without introducing a major error one can assume that $\xi \approx 1$.

Thus the size of condensations after contraction is determined by their rotation at the initial instant. To calculate the rotation of condensations we can turn to the premises regarding their genesis which we outlined above while evaluating their masses. According to the first premise, the angular momentum of the region from which the condensation evolved (and every volume element of which is in unperturbed circular Kepler motion around the Sun) with reference to the center of the condensation can be taken to be the mean rotational momentum of the condensation. If that region is a flat uniform circle, then its mean angular velocity of rotation around the center will be

$$\omega_0 = \frac{1}{2}\,\mathrm{rot}\,V = \frac{1}{2R}\frac{d}{dR}(VR) = \frac{1}{4}\,\omega_e, \tag{9}$$

where V and ω_e are the linear and angular velocities of revolution around the Sun. If instead of a circle one takes a uniform spherical region, the coefficient will be 1/5 instead of 1/4 (Artem'ev, 1963). The mean rotation of the region is direct, i.e., in the direction of revolution around the Sun. The direct rotation of this region had already been inferred by Prey (1920) and, based on a more accurate analysis, by Rein (1934). To obtain the angular momentum of a noncircular flat condensation, it is necessary to integrate the relationship

$$x^2\frac{V_0}{R_0} + y^2\left(\frac{dV}{dR_0}\right) = \omega_e\left(x^2 - \frac{1}{2}y^2\right) \tag{10}$$

over this entire region; the above relationship describes the angular momentum density with reference to the center of the condensation of an element situated at the point (x, y) and moving along a circular orbit around the Sun with velocity $V(R)$ (Edgeworth, 1949). The (a, b) frame of reference is nonrotating. Its origin lies at the center of the condensation. At the instant under consideration, the x and y axes lie along the orbit (in the direction of revolution) and along the radius vector, respectively. For an ellipse with semiaxes (x, y) along x and y, the mean angular velocity is given by

$$\omega_0 = \frac{2a^2 - b^2}{2(a^2 + b^2)}\,\omega_e = \alpha\omega_e. \tag{11}$$

The condition that the ratios of the kinetic to the potential energy at the ends of the axes be equal leads to $b/a = 3/4$, while the condition that the total particle energies be equal leads to $b/a = 1/2$. This gives α equal to 0.5 and 0.7, respectively.

If the region that is contracting inside the Sun's gravitational field is small compared with the distance R from the Sun and symmetric with reference to the x or y axis, then its angular momentum will be conserved (Hoyle, 1946). This allows us to use expression (8) to evaluate the radius of the condensation (1) after initial contraction. Assuming that $\alpha \approx 1/2$ and $\xi = 1$ and taking m_0 and r_0 according to (1), we obtain

$$\frac{r_1}{r_0} = \frac{\alpha^2}{\xi}\frac{r_0^3}{m_0}\frac{\omega_e^2}{G} = \frac{3}{\pi}\frac{\alpha^2}{\xi} \approx \frac{1}{4}. \tag{12}$$

The rotation of a condensation formed in the disintegration of a ring is particularly easy to estimate for $n_1 = 4$. In this case the condensation will rotate as a rigid body with angular velocity ω_v of revolution. Therefore in expression (12) we now have $\alpha \approx 1$. From (4') the condensation mass m is roughly three times greater than m_0, taken in (12) in accordance with (1). Therefore $r_1/r_0 \approx 1/3$ and not 1/4 as in (12).

Thus the initial contraction of the condensation causes its initial radius to shrink by a factor of three or four and its density to increase by one order or more.

19. Evolution of dust condensations

In the next phase the evolution of the condensations was considerably slower. In Edgeworth's view, it was determined by the tidal forces of the Sun which gradually slowed the condensation's rotation, thereby enabling it to contract along r. Due to the smallness of the condensations, however, the time span of this type of evolution would have been very considerable. The condensations must have contracted far more rapidly by collision and fusion. Thus when two condensations that have collided centrally combine, their mass doubles, while the angular momentum density remains as before. From (8), the radius of such an aggregate should shrink by a half and its density increase 16 times. For such rapid evolution the influence of the Sun's tidal forces on the rotation of the condensations would be negligible.

According to Gurevich and Lebedinskii, initially the condensations traveled along nearly circular orbits. Since only those condensations that lay along nearly the same orbit could have combined, aggregation (in the authors' view) kept the angular momentum density constant. The aggregation process lasted over the period P/e (i.e., of the order of 10^5 years in the Jupiter zone) and led to the formation of "secondary condensations" with masses of the order of $10^4 - 10^6$ times the masses of the primary condensations. Subsequently the relative velocities of the condensations were determined by their gravitational interaction in close encounters and were of the order of

$$v = \sqrt{\frac{Gm}{2r}}. \tag{13}$$

However, in the aggregation of condensations traveling initially along circular orbits, their angular momentum density with reference to the center of the condensation is not conserved. Mutual attraction deflects them from the circular orbits, and in collisions, depending on the initial difference ΔR in distance from the Sun, they acquire either a negative or a positive angular momentum; the change in angular momentum may even exceed the condensation's angular momentum prior to collision (Safronov, 1960b). Thus neither conservation of the angular momentum density of the aggregated condensations nor reverse rotation can be inferred to be the inevitable result of such aggregation — as often stated in discussions of Laplace's theory. At first, aggregation takes place in a narrow zone along the orbit; but due to encounters their relative velocities reach the values (13), increasing with the mass. The zones that feed the condensations expand

correspondingly, well before all the material in the narrow zone along the orbit has combined. In view of the continuity of the process of growth of the masses and orbital eccentricities of the condensations, as well as the absence of qualitative differences between the initial and subsequent stages of growth (especially where the acquisition of angular momentum is concerned), there is no justification for introducing the two concepts of "primary" and "secondary" condensations.

If a condensation of mass m and radius r combines with another condensation of mass m' and radius r', the latter will impart its own orbital angular momentum K_2 with reference to the center of the condensation m and angular momentum K_3, related to the spin. The orbital angular momentum is determined by the relative velocity v before impact and the impact parameter βr:

$$K_2 = \beta r v m'. \tag{14}$$

From (7), the angular momentum K_3 of the condensation, related to its spin, is given by

$$K_3 = \frac{2}{5}\mu\omega m' r'^2 = \frac{2}{5}\mu\sqrt{\lambda G m' r'}\, m', \tag{15}$$

where μ is the inhomogeneity coefficient.

Since the plane of the relative orbit m' can be inclined in any direction, the vector $\mathbf{K}_2$ can have any direction. The direction of the vector $\mathbf{K}_3$ is correlated, if at all, with the direction of the vector of the total angular momentum of the dust layer only at the beginning. After two or three collisions, the direction of $\mathbf{K}_3$ becomes random. In the process of aggregation, therefore, the vectors $\mathbf{K}_2$ and $\mathbf{K}_3$ add as random variables. Aside from the randomly directed components $\mathbf{K}_2$ and $\mathbf{K}_3$ of the angular momentum, during aggregation the systematic component $\mathbf{K}_1$ is also acquired; this is related to the general revolution of the entire system of condensations around the Sun and lies at right angles to the central plane of the system. Qualitatively the component $\mathbf{K}_1$ is of the same nature as the initial direct rotation of a condensation formed from the diffuse material of the revolving layer, which we estimated earlier with the aid of (9) and (10). It can also be regarded as a special result of the asymmetry of the impacts among the aggregating bodies (see Chapter 10). Then by analogy with (14) we can take

$$K_1 = \beta r v m'. \tag{16}$$

Assuming that the coefficient β does not depend on the mass of the growing body, its approximate numerical value can be found from the present rotation of the planets. We obtain $\beta \approx 0.04$.

If the masses m' combining with the condensation under consideration are very small compared with m, the random variables K_2 and K_3 will cancel each other out almost completely and the angular momentum of the condensation will be determined by its mean values as obtained by summing K_1. Writing the relative velocities in the form $v = \sqrt{\frac{Gm}{\theta r}}$, we obtain, by analogy with (13) and according to (7.15),

$$dK_1 = \frac{\beta}{\sqrt{\theta}} \sqrt{Gmr}\, dm = \frac{5\beta}{2\mu\sqrt{\xi\theta}} K_1 \frac{dm}{m}$$

and, consequently,

$$K_1 \propto m^{p_1}, \tag{17}$$

where $p_1 = \frac{5\beta}{2\mu\sqrt{\xi\theta}}$. Since $\mu\sqrt{\xi\theta} \approx 1$, $p_1 \approx 0.1$. For such a small exponent p_1, contraction of the condensation should be rapid. Indeed, from (15), the condensation radius

$$r \propto \frac{K^2}{m^3} \propto m^{2p_1-3},$$

and its density

$$\rho \propto \frac{m}{r^3} \propto m^{10-6p_1}.$$

Therefore it is sufficient that the mass increases by one order of magnitude for the condensation to contract to the state of a solid body with $\rho \approx 1$.

A different result is obtained for the aggregation of condensations of similar mass. The randomly oriented angular momenta $\mathbf{K}_2$ and $\mathbf{K}_3$ imparted to each condensation are of the same order as the angular momentum of the condensation under consideration and considerably larger than the mean value K_1, which can be disregarded. The resultant angular momentum is then determined by the probable deviation from this mean, i. e.,

$$K^2 = \sum (K_2^2 + K_3^2). \tag{18}$$

Here the direction of the vector $\mathbf{K}$ is arbitrary. Let us now evaluate K_2 and K_3. Let l_0 be the maximum impact parameter for which aggregation of colliding condensations still takes place. The mean value $\overline{l^2}$ in the interval between 0 and l_0 is given by

$$\overline{l^2} = \frac{1}{\pi l_0^2} \int_0^{l_0} l^2 2\pi l dl = \frac{1}{2} l_0^2.$$

From the well-known relation in the two-body problem, we have the following relation between the impact parameter l_0 and the distance r_0 at the instant of closest approach:

$$l_0^2 = r_0^2 \left[1 + \frac{2G(m+m')}{v^2 r_0}\right].$$

For $v^2 = \frac{Gm}{\theta r}$, where θ is of the order of a few units, the first term on the right is small compared with the second and can be disregarded. One may assume that for $r_0 > r$ it would be difficult for the condensations to combine, since the impacts are nearly tangential. Let us take $r_0 = \beta_0 r$, where $\beta_0 \approx 1$. Then

$$K_2^2 = \overline{l^2} v^2 m'^2 \approx G(m+m')\, m'^2 \beta_0 r = 2p_2 K^2 \frac{m'}{m}, \tag{19}$$

where

$$p_2 = \frac{25\rho_0}{8\xi\mu^2}\left(1 + \frac{m'}{m}\right)\frac{m'}{m}. \tag{19'}$$

In view of (15) one can write

$$K_3^2 = 2p_3K^2\frac{m'}{m}, \tag{20}$$

where

$$p_3 = \frac{m'^2r'}{2m^2r}. \tag{20'}$$

The quantities p_2 and p_3 depend on the ratio m'/m. If we take some function for the mass distribution of the condensations, we can find the mean values of p_2 and p_3 by simple integration. Then from (18) the mathematical expectation of probable increase of the angular momentum of the condensation with increasing mass can be written as

$$\frac{dK^2}{K^2} = 2(p_2 + p_3)\frac{dm}{m},$$

whence we obtain

$$K \propto m^{p_2+p_3}. \tag{21}$$

For the density of the condensation we derive

$$\rho \propto m^{10-8p_2-6p_3}. \tag{22}$$

For the condensations to contract and not dissipate during aggregation, it is therefore necessary that

$$p_2 + p_3 < \frac{5}{3}. \tag{23}$$

If the condensations are all identical this condition is not met ($p_2 \gg 1$). On the other hand, if the mass distribution of the condensations is such that $m' \ll m$, then p_2 and p_3 become smaller than p_1 and (17) holds for K. As we have already seen, in this case aggregation of the condensations leads to very rapid contraction.

It should be stressed that (21) merely gives the probable angular momentum. Qualitatively the evolution of the condensations is the following. Near-central impacts impart limited angular momentum and are accompanied by rapid contraction: when two identical condensations combine, the density increases by one whole order. Near-tangential impacts, on the contrary, impart a large angular momentum and hinder contraction. Thus the dispersion of densities and sizes increases very rapidly from the start. Since the unfavorable peripheral impacts have relatively little effect on the central part of the condensation, one should expect a gradual increase in the concentration of matter toward the center of the condensation. This means that the central parts of most condensations would grow solid at a fairly early stage, whereas the peripheral parts might remain in a diffuse state a fairly

long time. Transformation into solids takes place most rapidly in the case of the most massive and densest condensations. The exponent for the latter in (21) is small due to the fact that the effective m' is considerably smaller than m, first because m is maximum, and second because they can easily cut across diffuse condensations, acquiring from them only whatever material is swept away directly by their cross section and imparts very little angular momentum. It is sufficient to take $m'={}^1/_4 m$ for condition (23) to be met. In this case the conversion of condensations into solid bodies will take place when the mass has increased by a factor of 10^2 at the Earth's distance (10^3 at Jupiter's distance).

It is probable that the least massive condensations are unstable. For these $m'\approx m$ and, according to (19), $p_2\approx 6$. The aggregation of such condensations will lead, on the average, to a reduction rather than an increase in density. However, if a significant fraction of the cloud material remained in the diffuse state, not all of it entering into the composition of the condensations, then from (17) even the least massive condensations could have contracted efficiently by assimilating this scattered material of low angular momentum.

Taking a certain mean value $p=p_2+p_3$ in expression (21) for the angular momentum of the condensation, one can evaluate the duration of the condensation's evolution from the ordinary growth formula

$$\frac{dm}{dt}=\pi l_0^2\delta v\approx\frac{2}{3}\pi l_0^2\rho_0 v\approx\frac{2\pi}{3}\frac{\sigma}{H}vr^2\left(1+\frac{2Gm}{v^2r}\right).$$

Next, setting $K\propto m^p$, we have $r\propto m^{2p-3}$. For $v=\sqrt{Gm/\theta r}$ and in view of (12) we obtain

$$m^{6-4p}dm\approx\frac{2\pi(1+2\theta)}{3}\frac{\alpha^4}{\xi^2}r_0^2m_0^{6-4p}\sigma\frac{v}{H}dt. \tag{24}$$

The layer thickness H is related to $v=\sqrt{3\mathfrak{R}T/\mu}$ by (5.17). The gravitational attraction of a single cluster would give $H\propto v^2$, and that of the Sun $H\propto v$. At the start, when the density of the cluster is nearly critical, $H\propto v^{1.7}$. When m, and correspondingly v, increases several times (which takes place within fewer than a hundred revolutions), H increases significantly and the z component of solar gravitation increases. Quantity H then becomes nearly proportional to v and, from (3.5), is given by $Pv/4$, where P is the period of revolution around the Sun. Integrating (24) and introducing $m_0/r_0^2\approx 4\sigma$, in accordance with (6), we obtain the time required for the condensation mass to increase from m_1, at which the relation (3.5) becomes applicable, to m:

$$t-t_1\approx\frac{\xi^2}{4\alpha^4(7-4p)(1+2\theta)}\left[\left(\frac{m}{m_0}\right)^{7-4p}-\left(\frac{m_1}{m_0}\right)^{7-4p}\right]P. \tag{25}$$

Thus the time during which the mass increases from m_0 to m proves to be of the order of $(m/m_0)^{7-4p}P$. Since $\rho\propto m^{10-6p}$, the time during which the condensations transform into solid bodies will be of the order of

$$T\approx\rho_1^{-\frac{7-4p}{10-6p}}P, \tag{26}$$

where ρ_1 is the condensation's density after initial contraction, i.e., roughly one order greater than the Roche density. For $\bar{m}' = m/4$, the case discussed above, one obtains $\bar{p} \approx 1.2$ and the time in which the condensations convert into solids proves to be of the order of 10^4 years at the distance of the Earth and 10^6 years at the distance of Jupiter. The time required for evolution and transformation into solids may vary widely from one condensation to another, but on the whole the entire system of condensations converted within a cosmogonically short time into a cluster of solid bodies. The formation of numerous bodies hastened the break-up of condensations lagging behind in their development. Among the planets of the Earth group, the condensations transformed into solid bodies much sooner and with much smaller masses, on the average, than in the region of giant planets.

CONCLUSIONS

In early works by Shmidt (prior to 1950) it was assumed that solid bodies large enough to hold the particles falling into them were present in the protoplanetary cloud from the beginning. In an important contribution to the advance of the study of the early evolution of the protoplanetary cloud, Gurevich and Lebedinskii demonstrated that the cluster of solid bodies was formed as a result of the flattening of the dust layer enveloping the Sun and its disintegration into numerous condensations.

In Chapters 2 and 3 it was shown that the dust layer supposed by Gurevich and Lebedinskii to have constituted the primordial state is the natural result of the evolution of a gas-and-dust cloud of cosmic composition revolving around the Sun. It was found that the protoplanetary cloud was stable under small perturbations. The emergence of convection inside the cloud would require a very steep temperature gradient, such as could not have been achieved inside it. Calculation of the gravitational energy released by the cloud when interaction among turbulent eddies caused them to move closer to the Sun revealed that this energy was not large enough to maintain turbulence inside the cloud. Primordial random macroscopic motions present in the cloud must therefore have died away rapidly. This caused the solid material to separate out from the gaseous material. Dust particles began to settle toward the central plane of the cloud, forming there a layer of high density. The break-up of the dust layer was examined in detail in Chapter 5 on the basis of the theory of gravitational instability. The aggregation of the numerous dust condensations formed as a result of the break-up of the layer (see Chapter 6) led to the formation of a cluster of solid bodies. The subsequent evolution of the cluster and the formation inside it of the planets are discussed in Part II of this book. In Chapter 3 it was shown that gravitational instability was probably not present in the dust layer in the portion of the cloud adjacent to the Sun; this is because the high degree of flattening of the layer necessary for its presence could not have been achieved owing to the perturbation entering this zone from the evolving active Sun. In this zone the growth of solid bodies must have resulted from the aggregation of particles in collisions. Analysis of temperature conditions in the protoplanetary cloud (Chapter 4) indicated that hydrogen could not have been present in the solid state in the region of giant planets and that the planets rich in free hydrogen — Jupiter and Saturn — must have acquired it in the gaseous form by accretion in the concluding phases of growth.

Thus the principal stages in the evolution of the protoplanetary cloud enveloping the Sun are becoming increasingly clear and precise. The problem of the origin of the cloud itself, however, is still unsolved. From

our review in Chapter 1 of present-day theories regarding its origin, it is seen that they are all beset by considerable difficulties. The most promising theories at present seem to be those that envisage a common formation of Sun and protoplanetary cloud.

The urgent tasks today are to construct a consistent picture of the formation of the protoplanetary cloud enveloping the Sun and to study its physico-chemical evolution.

Part II

ACCUMULATION OF THE EARTH AND PLANETS

Chapter 7

VELOCITY DISPERSION IN A ROTATING SYSTEM OF GRAVITATING BODIES WITH INELASTIC COLLISIONS

20. Velocity dispersion in a system of solid bodies of equal mass

The process of planetary accumulation consists mainly of the collisions among and aggregation of numerous protoplanetary bodies. The relative velocity of these bodies is one of its major characteristics, since it determines the rate of planetary growth and the degree of fragmentation of the colliding bodies. There is a close relation between the relative velocities of the bodies and their size distribution. Initially the bodies, formed inside a flat dust disk, moved along nearly circular orbits and had low relative velocities. With aggregation and increasing mass, however, their gravitational interaction increased, as did the relative velocities and, correspondingly, the orbital eccentricities.

An approximate expression for the velocity dispersion in a system of protoplanetary bodies of equal mass was obtained by Gurevich and Lebedinskii (1950). The time of encounter of the bodies is many times smaller than the time required for one revolution around the Sun. Encounter can therefore be treated as in the two-body problem: the relative velocity vector v of approaching bodies of mass m does not change in magnitude but merely turns through the angle $\psi \approx Gm/Dv^2$, where $\psi \ll \pi/2$. In the process a change takes place in the eccentricity e of the body's orbit, given roughly by the expression

$$e\Delta e \approx \frac{Gm}{DV_c^2}, \tag{1}$$

where V_c is the circular velocity and D the impact parameter. Next the authors assumed that for very close encounters where $D = 2r$ (r being the radius of the body), the orbital eccentricity increment Δe is of the same order of magnitude as e itself. One can then find e from (1), and consequently the relative velocities

$$v = eV_c \approx \sqrt{\frac{Gm}{2r}}. \tag{2}$$

The result is correct in substance, but the expression for v needs to be refined. Relation (1) holds only for small ψ, i. e., for "distant" encounters. Since $\frac{\Delta e}{e} \approx \frac{|\Delta v|}{v} \approx \psi$, it should follow that $\Delta e \ll e$. But the authors applied

expression (1) to close encounters, taking $D=2r$ and presupposing that $\Delta e \approx e$. It is not clear what degree of error this produces, as for large ψ the relations become more complicated and fail to yield an expression similar to (2) for ν.

In a system with differential rotation the dispersion of velocities of the gravitating bodies is caused by the conversion of the energy of ordered motion into energy of random motion. The former is renewed in turn by the potential energy of the system with reference to the central mass – the system is somewhat compressed at right angles to the axis of rotation. If the collisions between the bodies were absolutely elastic, their velocities would increase steadily and no relation of the type (2) could hold. In a real system with inelastic collisions the dispersion of velocities is determined by the balance between the energy acquired in encounters and the energy lost in collisions. The assumption that $\Delta e \approx e$ for $D=2r$ (Gurevich and Lebedinskii) is essentially an implicit expression of this balance. The important "characteristic" dimension should indeed be of the order of $2r$. However, expression (2) does not tell us how the velocity dispersion depends on the nature of the collisions and on the degree of their inelasticity. The author has therefore carried out a more detailed analysis of the problem with a view to obtaining this dependence in an explicit form (Safronov, 1962d).

a) **Dispersion of velocities for small mean free paths.** Consider a rotating system of identical bodies of mass m and radius r not containing any gas. As long as the mean free path of the bodies is short compared with the distance from the Sun (i. e., the bodies themselves are small), the increase in velocity dispersion can be estimated from the ordinary hydrodynamic formulas for the dissipation of the energy of mechanical motion of a fluid due to viscosity. In an axially symmetric flow with angular velocity $\omega(R)$, the amount of energy dissipating per cm^3 per sec due to molecular viscosity is given by the expression (Lamb, 1932)

$$E=\eta R^2\left(\frac{d\omega}{dR}\right)^2=\frac{1}{3}\frac{\rho}{\tau}\lambda^2R^2\left(\frac{d\omega}{dR}\right)^2, \qquad (3)$$

where $\eta \approx 1/3\ \rho v\lambda$ is the coefficient of viscosity and R the distance from the axis of rotation. Applied to the system under consideration, E is the amount of energy of ordered (rotational) motion of the system that converts into energy of random motion. The above relation can be given a simple physical interpretation. On the average one third of all the particles move in a radial direction. Within the mean free time τ, particles traversing the mean free path λ acquire the relative velocity of differential motion $\Delta v=R\left(\frac{d\omega}{dR}\right)\lambda$, which changes from ordered to chaotic. The thermal energy $\frac{1}{2}\Delta v^2=\frac{1}{2}R^2\left(\frac{d\omega}{dR}\right)^2\overline{\lambda^2}$ is generated per unit mass within the time $\tau=\lambda/v$. Since $\overline{\lambda^2}=2\bar{\lambda}^2=2\lambda^2$, dividing this expression by τ and multiplying by $\rho/3$ (where ρ is the density of the medium) we recover (3).

There is little gravitational interaction between small bodies. But if the relative velocities are also small, the gravitational attraction of the bodies will need to be taken into account. Let τ_s be the time between two successive collisions of a body with other bodies and τ_g the time between successive close encounters involving substantial energy transfer. This occurs when the relative velocity vector of the body turns through an angle $\sim\pi/2$.

From (3), elastic collisions or close encounters will cause each body to acquire an average energy of relative motion $R^2\lambda^2\omega'^2/3\tau_g$ per sec per unit mass. Inelastic collisions are less effective in this respect. When colliding bodies aggregate, the relative velocity vector deflects from its initial direction by only about $\pi/4$ on the average. We can therefore assume that the increase in the energy of relative motion during collisions among bodies amounts to $\zeta_1 R^2\lambda^2\omega'^2/3\tau_s$ per sec per unit mass, where $\zeta_1 \leqslant 1$. Then

$$\varepsilon_1 = E/\rho = \frac{1}{3}\left(\frac{1}{\tau_g} + \frac{\zeta_1}{\tau_s}\right) R^2\lambda^2 \left(\frac{d\omega}{dR}\right)^2, \tag{4}$$

and the mean free time τ within which v turns through $\pi/2$ is given by

$$\frac{1}{\tau} = \frac{1}{\tau_g} + \frac{\zeta_1}{\tau_s}. \tag{5}$$

On the other hand, over each time interval τ_s the body will lose part of its own energy due to inelastic collision. Let the energy $\varepsilon_2\tau_s$ lost per unit mass amount to the fraction ζ of the kinetic energy of the body at impact. Let us denote by v_1 the velocity of the body (relative to the circular velocity) after a previous collision and by v_2 its velocity before the next collision. Then

$$v_2^2 = v_1^2 + 2\varepsilon_1\tau_s, \quad 2\varepsilon_2\tau_s = \zeta v_2^2. \tag{6}$$

If v^2 is the mean of v_1^2 and v_2^2, then

$$\varepsilon_2\tau_s = \frac{\zeta}{2}(v^2 + \varepsilon_1\tau_s). \tag{7}$$

Here v denotes the root mean square velocity. We take $v^2 = \frac{3\pi}{8}\bar{v}^2$ as for the Maxwellian distribution. As a result of the combined action of both effects, the body acquires the following amount of energy per unit mass per second:

$$\varepsilon = \varepsilon_1 - \varepsilon_2 = \frac{\zeta v^2}{2\tau_s}\left[\frac{2-\zeta}{3\zeta}\frac{R^2\lambda^2}{v^2}\left(\frac{d\omega}{dR}\right)^2\left(\frac{\tau_s}{\tau_g} + \zeta_1\right) - 1\right] = \frac{\zeta v^2}{2\tau_s}\left[\frac{(2-\zeta)\,2\omega^2\tau_s^2}{\pi\zeta(\tau_s/\tau_g + \zeta_1)} - 1\right]. \tag{8}$$

The geometrical collision cross-section of two bodies of radius r is $4\pi r^2$. But due to the low efficiency of near-tangential collisions, it is actually smaller, and we shall designate it by $\xi\pi r^2$. Due to gravitational attraction the collision cross-section increases $(1 + 2Gm/V^2r)$ times, where V is the relative velocity of the colliding bodies before the encounter. It can be assumed that on the average $V^2 = v^2 + v_2^2$. From (6) and (7) we obtain

$$v_2^2 = v^2\frac{2}{2 - k\zeta}, \tag{9}$$

where

$$k = \varepsilon_1/\varepsilon_2. \tag{10}$$

Therefore the mean free time between two successive collisions is given by

$$\tau_s = \frac{4\delta r}{3\sqrt{2}\,\xi\rho v\left[1 + \frac{Gm}{v^2 r}\left(\frac{1 - k\zeta/2}{1 - k\zeta/4}\right)\right]} = \frac{(\delta\theta/G)^{1/2}}{2\xi\rho\,(1 + a\theta)}, \tag{11}$$

where for convenience we have set

$$v^2 = \frac{Gm}{\theta r}, \tag{12}$$

$$\frac{1 - k\zeta/2}{1 - k\zeta/4} = a. \tag{13}$$

The time τ_g between encounters can be assumed to be equal to the relaxation time T_E or T_D, after Chandrasekhar (1942),

$$\tau_g \approx T_E = \frac{1}{16}\sqrt{\frac{3}{\pi}}\,\frac{(v^2)^{3/2}}{G^2 m\rho \ln(1 + D_0 v^2/2Gm)} = \frac{(\delta/G)^{1/2}}{8\rho\theta^{3/2}\ln(1 + D_0/2\theta r)}, \tag{14}$$

where D_0 is the mean distance between the bodies. Consequently,

$$\frac{\tau_s}{\tau_g} = 4\,\frac{\theta^2 \ln(1 + D_0/2\theta r)}{\xi(1 + a\theta)}. \tag{15}$$

Quantity D_0 can be expressed in terms of the number n of bodies per unit volume (Chandrasekhar, 1943):

$$D_0 \approx 0.554 n^{-1/3}. \tag{16}$$

According to (3.5)

$$\rho\vartheta = \frac{4\sigma}{P}. \tag{17}$$

Therefore

$$n = \rho/m = \frac{4\sigma}{P\vartheta m}. \tag{18}$$

Substituting for v from (12) and carrying out some simple operations, we obtain

$$\frac{D_0}{r} \approx \sqrt[3]{\frac{P\delta r}{\sigma}\sqrt{\frac{G\delta}{\theta}}}. \tag{19}$$

Eliminating the density ρ from (11) with the aid of (17), we find that

$$\tau_s = \frac{4\delta r}{3\sqrt{2}\,\xi\sigma(1 + a\theta)} \cdot \frac{P}{4}. \tag{20}$$

The condition that the foregoing relations be applicable for ε (short mean free paths) can be written as $\tau_s < P/4$. This gives $r < \xi\sigma(1 + a\theta)/\delta \sim 10$ cm for the terrestrial zone. On the other hand, the cloud of particles which we are investigating cannot be in a state approaching gravitational instability. This imposes the condition that the relative particle velocities $v > 10$ cm/sec. Then, from (12), it is necessary that $\theta < 10^{-7}$. Introducing the foregoing expressions for τ_ε and τ_s/τ_g in (8) and setting $\theta \ll 1$, we obtain

$$\varepsilon = \frac{\zeta v^2}{2\tau_\varepsilon}\left(\frac{4\pi(2 - \zeta)\delta^2 r^2}{9\zeta\zeta_1\xi^2\sigma^2} - 1\right). \tag{21}$$

In the terrestrial zone $\varepsilon = 0$ for $r = r_c \approx 3-4$ cm. If $r < r_c$, then $\varepsilon < 0$ and the particle velocities decrease. During collisions particles aggregate and their size increases. Depending on the initial density of the cloud and the initial ratio r/r_c, either gravitational instability will develop inside the cloud or, before this can happen, r will increase to r_c, the particle velocities will begin to rise, and the onset of instability becomes impossible. The particle velocities and therefore also the uniform thickness of the layer decrease roughly in inverse proportion to the size of the particles. Therefore if initially $r/r_c < \rho/\rho_{cr}$ in order of magnitude, then gravitational instability will develop before r increases to r_c. If $r > r_c$, then $\varepsilon > 0$ and the particle velocities will increase. Their mean free path increases in the process and within a few periods of revolution the initial equation (3), and therefore the expressions for ε derived from it, ceases to be valid.

b) **Dispersion of velocities for large mean free paths.** There exists no satisfactory theory of the transport of matter and motion in revolving systems for large mean free paths. In Chapter 2 we noted that in Prandtl's semiempirical theory, which was developed for turbulent motion (mixing length comparable with the dimensions of the system), it is assumed that the shearing stresses are determined not by the angular velocity gradient (2.17) but by the angular momentum gradient (2.18). For Kepler rotation ($\omega \propto R^{-3/2}$ and $\omega R^2 \propto R^{1/2}$), this causes a threefold reduction in the shearing stresses and a ninefold reduction in the generated energy compared with expression (3). Since Prandtl's theory is also nonrigorous, we introduced the additional factor $\beta' < 1$ in the expression equivalent to (3).

Introducing β' in (3) and replacing λ^2 by $\frac{1}{2}\overline{\Delta R^2}$, we can write the expression for the energy of relative motion per unit mass acquired by the body within the mean free time as follows:

$$\varepsilon_1 \tau = \frac{\beta'}{6} R^2 \left(\frac{d\omega}{dR}\right)^2 \overline{\Delta R^2}. \tag{22}$$

Here $\overline{\Delta R^2}$ is the mean value of the square of the radial displacement of the body. For short mean free paths, $\overline{\Delta R^2} = \overline{\lambda^2} = 2\lambda^2$. For large λ the quantity $\overline{\Delta R^2}$ can be much smaller than $\overline{\lambda^2}$, as the maximum radial deviation ΔR_m does not depend on λ and is determined only by the orbital eccentricity ($\Delta R_m \sim eR$). Let us evaluate $\overline{\Delta R^2}$. For small orbital eccentricities e,

$$R' - R = \frac{p}{1 + e\cos\varphi} - R \approx a(1 - e\cos\varphi) - R. \tag{23}$$

For large λ the true anomaly φ during encounter can assume any value between 0 and 2π with nearly equal probability.

For bodies whose relative velocity is directed radially at the initial instant ($v = v_R$), $a = R$ and

$$R' - R \approx -Re\cos\varphi, \quad \overline{\Delta R^2} \approx R^2 e^2 \frac{1}{2\pi} \int_0^{2\pi} \cos^2\varphi\, d\varphi = \frac{1}{2} R^2 e^2. \tag{24}$$

For bodies with relative velocity along the orbit ($v = v_\varphi$), $a = R/(1 \pm e)$ and

$$R' - R \approx -Re(\pm 1 + \cos\varphi),$$

$$\overline{\Delta R^2} \approx R^2 e^2 \frac{1}{2\pi} \int_0^{2\pi} (\pm 1 + \cos\varphi)^2 \, d\varphi = \frac{3}{2} R^2 e^2. \tag{25}$$

For small mean free paths, bodies with relative velocities along the radius ($v = v_R$) contribute to ε_1 (expression (5)). For large λ bodies with relative velocities along the orbit ($v = v_\varphi$) contribute three times as much to ε_1. Taking the role of both groups into account, we can substitute $2R^2e^2$ in (22) for $\overline{\Delta R^2}$, in accordance with (24) and (25). As will be shown below (see (43)),

$$v_R \approx eV_e, \quad v_\varphi \approx \frac{e}{2} V_e, \tag{26}$$

where $V_e = \omega R$ is the circular velocity.

Assuming that the mean inclinations of the orbits are equal to their mean eccentricities, as is the case in the asteroid system, we obtain

$$v_z^2 = \frac{1}{2} v_R^2. \tag{27}$$

We will therefore take the mean square relative velocity to be

$$v^2 = \frac{1}{2}(v_R^2 + v_\varphi^2) + v_z^2 \approx \frac{9}{8} e^2 V_e^2. \tag{28}$$

Introducing $\overline{\Delta R^2} = 2R^2e^2$ and $\omega = \sqrt{GM/R^3}$ in (22), we obtain

$$\varepsilon_1 \tau = \frac{3}{4} \beta' e^2 V_e^2 = \frac{2}{3} \beta' v^2 = \beta'' v^2. \tag{29}$$

The energy ε_2 lost in collision can be determined from (7). In view of (5), the change in the energy of relative motion of the bodies per gram per second is given by

$$\varepsilon = \varepsilon_1 - \varepsilon_2 = \varepsilon_1\left(1 - \frac{\zeta}{2}\right) - \frac{\zeta v^2}{2\tau_s} = \frac{\zeta v^2}{2\tau_s}\left[\frac{(2-\zeta)\beta''}{\zeta}\left(\frac{\tau_s}{\tau_g} + \zeta_1\right) - 1\right]. \tag{30}$$

Introducing the expression for τ_s/τ_g from (15), we obtain

$$\varepsilon = \frac{\zeta v^2}{2\tau_s}\left[\left(\frac{2-\zeta}{\zeta}\right)\beta''\left\{\frac{4\theta^2 \ln(1 + D_0/2\theta r)}{\xi(1 + a\theta)} + \zeta_1\right\} - 1\right]. \tag{31}$$

In a system in which the bodies collide but do not aggregate (constant m and r), their relative velocities should tend to a certain "equilibrium" value which can be determined from the condition that $\varepsilon = 0$; this gives

$$\theta^2 = \frac{\xi[\zeta - \zeta_1(2-\zeta)\beta''](1 + a\theta)}{4(2-\zeta)\beta'' \ln(1 + D_0/2\theta r)}. \tag{32}$$

For smaller velocities the parameter θ, which is related to v as in (12), is greater than the equilibrium value (32), and therefore $\varepsilon > 0$ and v should increase. If the velocity is greater than the equilibrium value, θ will be smaller than (32) and $\varepsilon < 0$. In this case the velocities of the bodies should decrease.

For $\beta' = 0.2$ (see below), expression (32) will give $\theta \sim 1$. For $\zeta < \frac{2\beta''\zeta_1}{1+\beta''\zeta_1} \approx$ ≈ 0.2 the expression for θ is found to be imaginary. In this case the loss of energy of motion of the bodies in collisions does not compensate for the corresponding increase caused by gravitational interactions in the rotating system and the velocities of the bodies will increase indefinitely.

From the standpoint of the process of planetary accumulation, it is more interesting to study systems in which the bodies aggregate in collisions and whose masses increase. Let us suppose for simplicity that aggregation takes place in every collision and that the masses of all the bodies remain the same during the growth process. As before, we will seek to obtain an expression for v in the form (12). Then, differentiating with respect to the time for constant θ, we obtain

$$\varepsilon \equiv v\frac{dv}{dt} = \frac{1}{3}\frac{G}{\theta r}\frac{dm}{dt}. \tag{33}$$

The mass of the body will double within the time τ_ε:

$$\frac{dm}{dt}\tau_\varepsilon = m. \tag{34}$$

Consequently,

$$\varepsilon = \frac{1}{3}\frac{Gm}{\theta r \tau_\varepsilon} = \frac{v^2}{3\tau_\varepsilon} = \frac{\zeta v^2}{2\tau_\varepsilon}\cdot\left\{\frac{(2-\zeta)\beta''}{\zeta}\left[\frac{4\theta^2 \ln(1+D_0/2\theta r)}{\xi(1+a\theta)} + \zeta'\right] - 1\right\}. \tag{35}$$

For this value of ε,

$$k = \frac{\varepsilon_1}{\varepsilon_2} = \frac{\varepsilon}{\varepsilon_2} + 1 = \frac{3}{4} + \frac{1}{2\zeta}, \qquad a = \frac{12-6\zeta}{14-3\zeta}. \tag{36}$$

From (35) we obtain the following expression for θ:

$$\theta^2 = \frac{2+3\zeta-3\beta''(2-\zeta)\zeta_1}{12(2-\zeta)\beta''\ln(1+D_0/2\theta r)}\,\xi(1+a\theta). \tag{37}$$

Quantity θ depends very weakly (logarithmically) on the ratio D_0/r which, in turn, depends weakly (as the cube root) on r. Relation (12) is therefore a physically valid expression of the dependence of the relative velocities of the bodies on their masses and radii. The quantity θ in this expression can be regarded as practically constant. Relation (12) can be seen as a generalization of formula (2) of Gurevich and Lebedinskii, as it constains the parameter θ, which is dependent on the properties of the system of bodies under consideration.

When bodies combine in collisions, it is not difficult to evaluate the parameter ζ characterizing the degree of inelasticity of the collisions. Let two bodies, each of mass m, have the same velocity v (in magnitude) of relatively Keplerian circular motion and let ψ be the angle between the vectors v. The velocity v' after collision and aggregation can be determined from the condition that the angular momentum is conserved:

$$2mv' = 2mv\cos\frac{\psi}{2}, \qquad v' = v\cos\frac{\psi}{2}. \tag{38}$$

From the definition of ζ,

$$\zeta v^2 = v^2 - v'^2 \text{ and } \zeta = \sin^2\frac{\psi}{2}. \tag{39}$$

The collision frequency ν is proportional to the relative velocity V of the bodies and to the effective collision cross-section l^2:

$$\nu(\psi) \propto Vl^2 \propto 2v \sin\frac{\psi}{2}\left(1 + \frac{2Gm}{r4v^2 \sin^2\frac{\psi}{2}}\right) \propto \sin\frac{\psi}{2} + \frac{\theta}{2\sin\frac{\psi}{2}}. \tag{40}$$

For a random distribution of the vectors v over the directions, the mean value of ζ is

$$\zeta = \frac{\int \zeta\nu d\Omega}{\int \nu d\Omega} = \frac{\int_0^\pi \sin^2\frac{\psi}{2}\left(\sin\frac{\psi}{2} + \frac{\theta}{2\sin\frac{\psi}{2}}\right)\sin\psi d\psi}{\int_0^\pi \left(\sin\frac{\psi}{2} + \frac{\theta}{2\sin\frac{\psi}{2}}\right)\sin\psi d\psi} = \frac{6 + 5\theta}{10 + 15\theta}. \tag{41}$$

Thus ζ in turn depends on θ, though relatively weakly: for $\theta = 1$, $\zeta = 0.44$ and for $\theta = 3$, $\zeta = 0.38$. Without introducing a major error, one can assume $\zeta = 0.4$ in expression (37) for θ.

In reality the velocity distribution of the bodies is not isotropic. Velocities in the radial direction are, on the average, twice the tangential velocities. However, this does not alter ζ substantially.

Earlier in the expression for τ we introduced the quantity $\zeta_1 < 1$, which is linked to the fact that in inelastic collisions the velocity vector of a body will turn through an angle smaller than $\pi/2$ on the average. The relaxation time T_s, according to Chandrasekhar, is defined as the time in which the sum of the squares of sines of the angles of deflection of the body in encounters reaches unity. In the case we are considering, in which colliding bodies aggregate, v is turned through the angle $\psi/2$. Therefore

$$\zeta_1 = \overline{\sin^2\frac{\psi}{2}} = \zeta. \tag{42}$$

Tables 6 lists the values of the parameter θ calculated from (37) for $\zeta = \zeta_1 = 0.4$, $\xi = 2$, $\delta = 2$ and $\beta' = 0.2$ for the Earth zone ($\sigma = 10\,g/cm^2$) and for the Jupiter zone ($\sigma = 20\,g/cm^2$), as a function of the radii r of the bodies and their densities δ.

TABLE 6

r, cm	δ, g/cm^3	θ	
		Earth zone	Jupiter zone
10^2	2	1.4	1.1
10^3	2	1.08	0.96
10^4	2	0.94	0.84
10^5	2.5	0.83	0.77
10^7	3.0	0.70	0.66

For $\xi = 4$ the values of θ come closer to $\theta = 2$, as adopted in expression (2) of Gurevich and Lebedinskii; but for fairly large bodies they are still perceptibly less.

21. Increase in energy of relative motion in encounters

In all the preceding calculations a significant error stemmed from the uncertainty in the parameter β' in (22), which was introduced in an attempt to draw an analogy between rotational motion for large mean free paths and turbulent rotational motion. As the theory of turbulence has so far failed to give a definite numerical value for this parameter, it is desirable to attempt to evaluate β' by another, more direct method.

In order to avoid complicating the problem excessively, let us consider the following idealized scheme. The body m_1 is traveling along an elliptical orbit of small eccentricity e in the central plane of a rotating system (perpendicular to the axis of rotation). Let the other bodies m_2 which it encounters move in the same plane along circular Kepler orbits to which they are, as it were, fastened, not deviating from them during encounters. This assumption indicates a kind of averaging of the results of the encounter of the body m_1 with other bodies of different velocities with respect to the circular velocity but directed with equal probabilities along different directions, so that the mean value of this relative velocity is zero. Since during encounters the velocity vector v_{1_g} of the body m_1 shifts with respect to the center of gravity of m_1 and m_2, the energy transferred will depend first and foremost on the magnitude of this velocity. The assumption that the relative velocity v_2 of the bodies m_2 is zero reduces v_{1_g}, but the coincidence of the center of gravity with m_2 increases v_{1_g}, so that the errors introduced by these two simplifications compensate each other to a large extent. From (29) it is evident that the expression $^2/_3\beta' v^3$ represents the mean increase in the relative energy of the body due to encounters within the relaxation time, i.e., for an average rotation of 90° in the direction of the relative velocity. We will therefore consider encounters of m_1 with m_2 at different points on the orbit of m_1 for which the vector $v_{1_g} = v_1$ rotates by 90°. We assume further that all the bodies have identical masses: $m_1 = m_2 = m$.

It can be shown that at the point of intersection between the elliptical orbit (semimajor axis a and eccentricity e) and the circular orbit (radius R), the relative velocity of the body on the elliptical orbit with respect to the one on the circular orbit is given by

$$v_1^2 = V_c^2\left[3 - \frac{R}{a} - 2\sqrt{\frac{a}{R}(1-e^2)}\right]. \tag{43}$$

Introducing

$$R = \frac{a(1-e^2)}{1 + e\cos\varphi}, \tag{44}$$

we obtain

$$v_1^2 = V_c^2\left(3 - \frac{1-e^2}{1+e\cos\varphi} - 2\sqrt{1+e\cos\varphi}\right) =$$
$$= e^2V_c^2\left[1 - \frac{3}{4}\cos^2\varphi - e\left(\cos\varphi - \frac{7}{8}\cos^2\varphi\right) + \ldots\right]. \qquad (45)$$

Up to first order infinitesimals in e, $v_1 = v_\varphi \approx \frac{1}{2}eV_c$ at perihelion and aphelion (true anomaly $\varphi = 0$ and 180°), and $v_1 = v_R \approx eV_c$ at the intermediate distance $R = a$ ($\varphi = 90$ and 270°), i. e., it becomes twice as large.

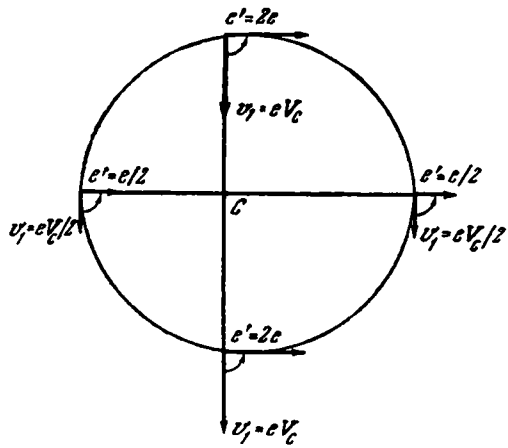

FIGURE 3. Change in orbital eccentricity of a body due to close encounters with other bodies at different points along the orbit. When the relative velocity vector $\mathbf{v}_1$ rotates through the angle $\pi/2$ without change of magnitude at perihelion and aphelion of the initial orbit, the eccentricity of the new orbit is half the initial value; when this takes place at the intermediate distance, it becomes twice the initial value.

The difference between the relative velocities v_1 at different points along the orbit is the main factor controlling the redistribution of velocities and energy during encounters. Bodies lying at perihelion or aphelion of their orbits ($v = v_\varphi$) will, after close encounter and rotation of v_1 by 90°, lie at the intermediate distance along the new orbit ($v = v_R$, Figure 3). But as the quantity v_1 remains the same, the orbital eccentricity decreases by a factor of two. If, on the other hand, encounter takes place at the intermediate distance ($v = v_R$), the 90° rotation in v_1 will be accompanied by a twofold increase in the orbital eccentricity. The energy of relative motion should be determined not by the value of v_1^2 at the given point but by the mean value $\overline{v_1^2}$ along the orbit:

$$\overline{v_1^2} \approx e^2V_c^2\left(1 - \frac{3}{4}\overline{\cos^2\varphi}\right) = \frac{5}{8}e^2V_c^2. \qquad (46)$$

i. e., by the square of the eccentricity, when e is small and higher order terms can be disregarded. Therefore the energy of relative motion will decrease by a factor of four in encounters at perihelion and aphelion and increase by a factor of four at the intermediate distance. On the average this energy increases, since $\frac{1}{2}\left(\frac{1}{4}v_1^2+4v_1^2\right)=\frac{17}{8}v_1^2>v_1^2$. This indeed constitutes the physical essence of the mechanism by which the dispersion in the velocities of bodies in a rotating system increases due to their gravitational interaction for large mean free paths.

Let us evaluate the increment in the energy of relative motion. By examining the two-body problem one can derive the following relations between the parameters R and V_c of the circular orbit, the parameters a and e of the elliptical orbit, the relative velocity v_1 at the point of intersection between them, which lies at the angular distance φ from perihelion, and the angle ψ between v_1 and V_c:

$$e^2=1-\frac{R}{a}\left(1+\frac{v_1}{V_c}\cos\psi\right)^2, \tag{47}$$

$$\frac{R}{a}=1-2\frac{v_1}{V_c}\cos\psi-\frac{v_1^2}{V_c^2}, \tag{48}$$

$$\frac{v_1^2}{V_c^2}\sin^2\psi=\frac{e^2\sin^2\varphi}{1+e\cos\varphi}. \tag{49}$$

Then correct to the infinitesimals e^2 and v_1^2/V_c^2

$$e^2\approx\frac{v_1^2}{V_c^2}(1+3\cos^2\psi). \tag{50}$$

After encounter, $v_1'=v_1$, $\psi'=\psi\pm\frac{\pi}{2}$, and therefore

$$\Delta e^2=e'^2-e^2=\frac{3v_1^2}{V_c^2}\left[\cos^2\left(\psi\pm\frac{\pi}{2}\right)-\cos^2\psi\right]=3\frac{v_1^2}{V_c^2}(2\sin^2\psi-1). \tag{51}$$

From (45) and (49) we obtain

$$\Delta e^2\approx 3e^2\left(1-\frac{5}{4}\cos^2\varphi\right). \tag{52}$$

In the two-dimensional problem under consideration, the frequency of close encounters $\nu(\varphi)$ is proportional to DV, where $V=v_1$ is the relative velocity of the approaching bodies and D the impact parameter for which the vector $\mathbf{v}_1$ will rotate by $\pi/2$. Then $D\propto V^{-2}$ and from (45)

$$\nu(\varphi)\propto 1/v_1\propto\left(1-\frac{3}{4}\cos^2\varphi\right)^{-1/2}. \tag{53}$$

The mean value of $\overline{\Delta e^2}$ for all points of the orbit is given by

$$\overline{\Delta e^2}=\int_0^{2\pi}\nu(\varphi)\,\Delta e^2 d\varphi=3e^2\frac{\int_0^{2\pi}\left(1-\frac{5}{4}\cos^2\varphi\right)\left(1-\frac{3}{4}\cos^2\varphi\right)^{-1/2}}{\int_0^{2\pi}\left(1-\frac{3}{4}\cos^2\varphi\right)^{-1/2}d\varphi}=0.8e^2. \tag{54}$$

For the spatial problem one should take $\nu(\varphi) \propto D^2 V$ and

$$V \approx \sqrt{v_1^2 + \overline{v_1^2}} = eV_e \sqrt{\frac{7}{8}\left(1 + \frac{6}{7}\sin^2\varphi\right)}. \tag{55}$$

In this case $\overline{\Delta e^2} \approx 0.69 e^2$, i. e., it differs only slightly from the above value.

From (29) and (46),

$$\varepsilon_1 \tau = \frac{3}{4}\beta' e^2 V_e^2 = \frac{\overline{\Delta v_1^2}}{2} = \frac{5}{16}\overline{\Delta e^2} V_e^2$$

and

$$\beta' = \frac{5}{12}\frac{\overline{\Delta e^2}}{e^2} \approx 0.3. \tag{56}$$

The result obtained applies to the motion of m_1 in a plane perpendicular to the axis of rotation, where encounters will alter the orbital eccentricity most effectively. In the general case of motion along inclined orbits, one should expect a smaller value of β'. For purposes of numerical evaluation we will assume that $\beta' = 0.2$.

We have confined ourselves here to close encounters $\left(\psi' - \psi = \frac{\pi}{2}\right)$. But it is known that distant encounters play no less important a role in the exchange of energy. As v_1 rotates through the angle $\Delta\psi$, the value of e^2 determined from (50) will change as follows:

$$\Delta e^2 = 3\frac{v_1^2}{V_e^2}[\cos^2(\psi + \Delta\psi) - \cos^2\psi] = 3\frac{v_1^2}{V_e^2}[\sin^2\Delta\psi(\sin^2\psi - \cos^2\psi) - \\ - 2\sin\Delta\psi\cos\Delta\psi\sin\psi\cos\psi]. \tag{57}$$

Replacing ψ by φ in accordance with (49) we obtain

$$\Delta e^2 = 3e^2\left[\sin^2\Delta\psi\left(\sin^2\varphi - \frac{1}{4}\cos^2\varphi\right) - \sin\Delta\psi\cos\Delta\psi\sin\varphi\cos\varphi\right]. \tag{58}$$

In view of the symmetry of $\nu(\varphi)$ (according to (53) and (55), $\nu(\varphi)$ is an even function of φ), when averaging over φ the term with $\sin\varphi\cos\varphi$ gives zero. The factor containing $\sin^2\Delta\psi$ represents the increment Δe^2 in close encounter ($\Delta\psi = \pi/2$), as given in (52). Therefore

$$\overline{\Delta e^2} = \sin^2\Delta\psi\,(\overline{\Delta e^2})_{\Delta\psi=\pi/2}. \tag{59}$$

Hence it is seen that the effect of many distant encounters will be the same as that of one close encounter if $\sum \sin^2\Delta\psi = 1$. But this is the condition that defines the relaxation time T_D, after Chandrasekhar, which is nearly the same as the expression T_E taken above for the time τ_g between encounters (see (14)). Thus when distant encounters are allowed for, the situation becomes quite satisfactory.

22. Velocity dispersion of bodies moving in a gas

In view of the fact that the gaseous component of the protoplanetary cloud (amounting originally to 99% of the mass) was not scattered at once, in the early phases of growth of the protoplanetary bodies the gas offered resistance to their motion. Hence there is a need to adjust our foregoing estimate of the velocities of the protoplanetary bodies.

The importance of friction in the gas is immediately apparent when one compares this effect with the deceleration which occurs during aggregation. Let ε_3 be the energy lost by a body due to the resistance of the gas per gram per second. Then, from (3.1),

$$\varepsilon_3 = Fv = cv^2, \qquad c = \frac{\rho_g v_g}{\delta r} \tag{60}$$

and, from (29) and (6),

$$\varepsilon_1 = \frac{\beta'' v^2}{\tau}, \qquad \varepsilon_2 = \frac{\zeta v_2^2}{2\tau_e}. \tag{61}$$

From (11), (17) and (60) we have

$$\frac{\varepsilon_3}{\varepsilon_2} = \frac{2c\tau_e v^2}{\zeta v_2^2} = \frac{8}{3\sqrt{2}\,\xi\zeta(1+a\theta)}\frac{v^2}{v_2^2}\frac{\sigma_g}{\sigma_p} \sim \frac{\sigma_g}{(1+a\theta)\sigma_p}. \tag{62}$$

At the initial stage of growth $\sigma_g/\sigma_p \sim 10^2$ and the deceleration due to friction is far more effective than that due to inelastic collisions. It is only when the mass of gas inside the cloud becomes less than the mass of solid material due to dissipation that the resistance of the gas can be disregarded.

From (60), (61), (9) and (35),

$$\varepsilon = \varepsilon_1 - \varepsilon_2 - \varepsilon_3 = \left(\frac{\beta''}{\tau} - \frac{\zeta}{\tau_e(2-k\zeta)} - c\right)v^2 = \frac{1}{3\tau_e}v^2, \tag{63}$$

where, as before, $k = \varepsilon/\varepsilon_2 + 1$ and is expressed in terms of ζ in the form (36). Let us determine τ_e/τ from this and compare it with its expression from (5) and (15):

$$\zeta_1 + \frac{\tau_e}{\tau_g} = \zeta_1 + \frac{4\theta^2 \ln(1 + D_0/2\theta r)}{\xi(1+a\theta)} = \frac{c\tau_e}{\beta''} + \frac{2+(3-k)\zeta}{3\beta''(2-k\zeta)}. \tag{64}$$

Eliminating $c\tau_e$ as in (62) and denoting the parameter θ for motion of the bodies in the gas by θ_g, we obtain

$$\theta_g^2 = \frac{1}{3\beta'' \ln(1 + D_0/2\theta_g r)}\left[\frac{1}{\sqrt{2}}\frac{\sigma_g}{\sigma_p} + \frac{2+(3-k)\zeta - 3\beta''\zeta_1(2-k\zeta)}{4(2-k\zeta)}\,\xi(1+a\theta_g)\right]. \tag{65}$$

The values of θ_g are cited in Table 7. The density δ of the bodies is assumed to increase with their size from 2 to 3 g/cm^3.

When the bodies are small (a few meters in diameter), θ_g is an order of magnitude greater than θ as computed in the absence of a gas. But even in a system of larger bodies (tens of kilometers in diameter) when the gas is retained θ_g will exceed θ by a factor of 4 to 5. The rate of growth of the protoplanetary bodies is proportional to $(1+2\theta)$. The presence of the gas

TABLE 7

Planet	σ_p, g/cm^2	σ_g/σ_p	r, cm				
			10^3	10^5	10^6	10^7	10^7
Earth . . .	10	200	43	22	15	12	9.1
"	10	70	16	10	7.8	6.4	5.1
Jupiter . .	20	70	10	8.0	6.7	5.7	4.8
Neptune . .	1	70	6.4	5.6	5.1	4.7	4.1

must therefore have substantially hastened the growth of the bodies. It also substantially reduced the effectiveness of fragmentation in collisions.

23. Velocity dispersion in a system of bodies of varying mass

When evaluating the velocity dispersion in a system of bodies of varying mass one meets additional difficulties stemming not only from the increasing complexity of the formulas but also from the necessity of knowing the size distribution function of the bodies. The latter must be determined from a complicated integrodifferential equation (see Chapter 8) whose solution, in turn, depends on knowledge of the velocities of the colliding bodies. The two problems should strictly speaking be solved simultaneously. But the complete problem is insoluble in analytic form and has to be split into two problems, one in which the distribution function of the bodies is assumed to be known and the other in which their relative velocities are assumed to be known.

In this book we will confine ourselves to an approximate estimation of the main factors governing the velocities of the bodies in the system, presupposing for simplicity that the size (mass) distribution of the bodies obeys a power law:

$$n(m)\,dm = cm^{-q}dm, \quad c = (2-q)\rho m_1^{q-2}, \tag{66}$$

where m_1 is the mass of the largest body in the distribution. The above expression for c is suitable for $q < 2$, which corresponds to $p = 3q - 2 < 4$ in the distribution $n(r) = c_1 r^{-p}$ over the radii.

We assume as before that a body of mass m' loses energy $\varepsilon_2 = \frac{\zeta' v^2}{2\tau_e}$ per second due to collisions. We determine τ_e as the time within which the body m' collides with bodies having total mass m' and, combining with them, doubles its mass. Consider the case $p > 3$ (i.e., $q > 5/3$) where τ_e is less than the "lifetime" of m', i.e., than the time between collisions among m' and larger bodies. Then the doubling of m' will take place thanks to the aggregation of smaller bodies $m'' < m'$. Since the shift in the direction of the velocity of m' in collisions with small bodies m'' is small, ζ_1 will be small and $\tau \approx \tau_e$. Frequent distant encounters and frequent collisions with smaller

bodies will cause a uniform energy generation ε_1 and energy absorption ε_2, so that the velocity v can be regarded as constant over the time τ_s. Therefore

$$\varepsilon_1\tau_g=\beta''v^2, \qquad \varepsilon_2\tau_s=\zeta'v^2/2. \tag{67}$$

Then, in view of (35),

$$\varepsilon=\varepsilon_1-\varepsilon_2=\left(\frac{\beta''}{\tau_g}-\frac{\zeta'}{2\tau_s}\right)v^2=\frac{v^2}{3\tau_{s0}}, \tag{68}$$

where τ_{s0} is the doubling time of the mass m of the largest body. Hence

$$\frac{\tau_s}{\tau_g}=\frac{\zeta'}{2\beta''}+\frac{1}{3\beta''}\frac{\tau_s}{\tau_{s0}}=b. \tag{69}$$

For a power law of mass distribution

$$\tau_s/\tau_{s0}=(m'/m)^{q-5/3}.$$

Since the ratio τ_s/τ_g is a function of θ, the latter can be determined from expression (69).

In Chapter 11 we will show that the masses of the bodies that fell into the Earth did not exceed 10^{-3} Earth masses. This means that in the concluding phases of growth the embryo Earth was far in advance of the other bodies and that it dropped out of the general mass distribution cm^{-q} of the bodies. Therefore two cases are to be considered: a) the initial stage of accumulation in which the planet embryo m is the largest body in the distribution m^{-q}, and b) the concluding stage in which the largest body m_1 in the distribution m^{-q} amounts to a small fraction ($\sim 10^{-3}$) of the mass of the planet embryo.

a) **Initial stage.** According to Chandrasekhar (1942), for a body of mass m' moving with velocity v in a system of bodies of mass m'', the relaxation time T_D is given by

$$T_D=\frac{v^3}{8\pi\mathcal{K}(x_0)Gm''^2n\ln\left(1+\frac{D_0v^2}{G(m'+m'')}\right)}, \tag{70}$$

where n is the number of bodies m'' per unit volume and $\mathcal{K}(x_0)$ is a function of the ratio of the velocity v of the body m' to the velocity dispersion of the bodies m''.

To obtain the relaxation time T'_D for a body m' moving in a system of bodies of varying mass, it is necessary to take the inverse of T_D (according to (70)) and integrate over all m'':

$$\frac{1}{T'_D}\approx\frac{8\pi G^2}{v^3}\overline{\mathcal{K}\ln\left(1+\frac{D_0v^2}{G(m'+m'')}\right)}\int_0^{m_1}m''^2n(m'')\,dm''. \tag{71}$$

Setting $m_1=m$ and $v^2=\frac{Gm}{\theta' r}$ and taking $n(m)$ in the form (66), we obtain

$$T'_D=\frac{3-q}{2-q}\frac{v^3}{8\pi f_1G^2m\rho}, \tag{72}$$

where we assume that $f_1 = \overline{\mathscr{H} \ln\left(1 + \frac{D_0 v^2}{G(m' + m'')}\right)}$.

The time τ_s within which the body m' collides with bodies $m'' < m'$ of total mass m' can be found by integrating the inverse of τ_s, given by expression (11), over m'':

$$\frac{1}{\tau_s} = \frac{\pi r'^2 f_2 v}{m'} \int_0^{m'} m'' n(m'')\, dm'' = \frac{\pi r'^2 f_2}{m'} \rho v \left(\frac{m'}{m}\right)^{2-q}, \tag{73}$$

where

$$f_2 = \overline{\sqrt{1 + \frac{v^{*2}}{v^2} \xi' \left(1 + \frac{2G(m' + m'')}{(r' + r'')V^2}\right)}}.$$

Introducing the values of T_D (instead of τ_g) and τ_s in (69) and assuming that $v^2 = \frac{Gm}{\theta' r}$, we obtain

$$\frac{\tau_s}{\tau_g} = \frac{2-q}{3-q} \frac{8 f_1}{f_2} \theta'^2 \left(\frac{m'}{m}\right)^{q - \frac{5}{3}} = b \tag{74}$$

and

$$\theta'^2 = \frac{3-q}{2-q} \frac{b f_2}{8 f_1} \left(\frac{m}{m'}\right)^{q - \frac{5}{3}}, \tag{75}$$

where b is defined by (69).

The values of θ' calculated from (75) for the Earth zone for $\zeta' = 0.7$ are listed in Table 8.

TABLE 8

r, cm	$p=3$			$p=3.5$			
	$r' = r$	$r' = 0.1r$	$r' = 0.01\,r$	$r' = r$	$r' = 0.1\,r$	$r' = 0.01\,r$	θ
10^3	3.5	1.2	1.2	6.4	2.6	5	1.5
10^5	2.3	1.1	1.0	4.0	2.2	4	1.1
10^7	1.9	0.9	0.9	2.9	2.0	3	0.9

The first column shows the radius r of the largest body in the planet's zone. The next columns list the values of θ' for bodies of radius r'. For $p = 3$, the smaller bodies have a smaller θ' (greater velocities). This is due to the relative decline in importance of collisions resulting from reduced gravitational focusing and the conversion of the collision cross-section into a geometrical cross-section. A similar effect is observed for $p = 3.5$, though only when r moves to $0.1\,r$. For smaller r' the parameter θ again increases

due to the high collision frequency, which increases progressively with decreasing r'. As the radius r of the largest body (and correspondingly of all other bodies) increases, θ' decreases: owing to the rising velocities the thickness of the system increases, with D_0/r and f_1 increasing correspondingly. The values of θ for a system of identical bodies of radius r are given for comparison in the last column. The relative velocities (in cm/sec) of the bodies corresponding to the values of θ' are given in Table 9.

TABLE 9

r, cm	$p=3$		$p=3.5$	
	$r'=r$	$r'\ll r$	$r'=r$	$r'\ll r$
10^3	0.4	0.7	0.3	0.3
10^5	50	80	40	40
10^7	6600	9600	5400	5300

In the early evolutionary phase of the system relative velocities were small, so that combinations were predominant in collisions, as is seen from the table. Fragmentation became significant only when the largest bodies grew to a diameter of several kilometers. Doubt has often been expressed in the cosmogonic literature as to whether the aggregation of small rigid bodies was possible. Thus in a review by Gold (1963) the formation of sufficiently large bodies capable of further growth by gravitational attraction is included among obscure and delicate problems of planetary cosmogony urgently requiring a solution. One way of approaching this particular problem was to develop a theory in which gravitational instability of the dust component inside the cloud led to the formation of sufficiently massive dust condensations that evolved eventually into bodies. The above evaluation of the relative velocities of the bodies shows that when conditions necessary for gravitational instability were absent, the bodies could have grown to a diameter of several kilometers by direct aggregation in collisions. Doubts as to the possibility of direct growth of the bodies at the early stage stem exclusively from the absence of quantitative estimates of the relative velocities of the bodies and from notions inherited from von Weizsäcker (though not justified in any way) regarding the prevalence of velocities of the order of 1 km/sec in the protoplanetary cloud.

b) **Concluding stage**, $m \gg m_1$. When the mass m of the planet embryo is of the same order as the mass of all the bodies in its zone, perturbations of the embryo m in the motions of other bodies become significant and it becomes necessary to consider these separately when evaluating the relaxation time τ_g''.

The inverse of the latter is now composed of $1/T_D'$, related to perturbations of all bodies other than m in the distribution m^{-q} up to the maximum mass $m_1 \sim 10^{-3} m$ (see Chapter 8), and $1/T_D^*$, related to perturbations of the embryo m. From (71) we have, in place of (72),

$$T_D^* = \frac{3-q}{2-q} \frac{v^3}{8\pi f_1 G^2 m_1 \rho''}, \tag{76}$$

where ρ'' is the density due to all bodies other than m.

To determine T_D^* it is necessary to evaluate the frequency of encounters between the body and the planet embryo at various distances from it. We will assume that the bodies move completely at random over the entire planetary zone, which we will treat as closed. This is facilitated both by mutual perturbations among the bodies and by perturbations emanating from the planet embryo. Öpik (1951) mentions, for instance, that under the influence of the Earth's secular perturbations the line along which the inclined orbit of the body intersects the plane of the ecliptic (nodal line) describes a complete turn only once in $6 \cdot 10^4$ years. Since $m' \ll m$, when evaluating the frequency of collision of the bodies m' with the planet embryo they can be treated as point masses. If a point is traveling in space with a velocity v and encounters on the average n orthogonally placed areas s per unit volume along its path, the mathematical expectation for the point entering this small area in the time t will be given by $nsvt$. We assume that in the zone adjacent to the planet, which for a body moving with velocity v has a volume

$$SH = \frac{Q}{\sigma_0}\frac{Pv}{4},$$

there exists a single planetary embryo. Setting $n = 1/SH$ and $s = 2\pi DdD$, we obtain the mathematical expectation of the number of passages of the body at an impact distance between D and $D + dD$ from the planet, given by

$$nsvt = \frac{8\pi\sigma_0 t DdD}{QP}. \tag{77}$$

Here σ_0 is the total surface density of matter in the zone, including the planet m. As the body approaches the planet its relative velocity vector v rotates through the angle Ψ:

$$\operatorname{tg}\frac{\Psi}{2} = \frac{Gm}{Dv^2} = \frac{1}{\sqrt{x}}, \quad \sin^2\Psi = \frac{4x}{(1+x)^2}. \tag{78}$$

The mathematical expectation for deflection Ψ within the time t for numerous encounters at various impact distances D is given by the expression

$$\sum \sin^2\Psi = \frac{8\pi\sigma_0 t}{QP}\int_{D_m}^{D_M}\frac{4x}{(1+x)^2}DdD = \frac{16\pi\sigma_0 t}{QP}\frac{G^2m^2}{v^4}f_3,$$

where

$$f_3 = \int_{x_m}^{x_M}\frac{xdx}{(1+x)^2} = \ln\frac{1+x_M}{1+x_m} - \frac{1}{1+x_m} + \frac{1}{1+x_M} \sim \ln x_M \approx 2\ln\frac{D_M}{\theta r}. \tag{79}$$

The relaxation time T_D^* is given by the condition

$$\sum \sin^2\Psi = 1. \tag{80}$$

Consequently

$$T_D^* = \frac{QPv^4}{16\pi a_0 G^2 m^2 f_3}. \tag{81}$$

If we take the collision cross-section $\pi r^2(1+2\theta'')$ in (77) instead of $2\pi DdD$, we obtain the mathematical expectation of the number of collisions between the body and the planet embryo within the time t. Setting it equal to unity, we obtain

$$\tau_e^* = \frac{QP}{4\pi a_0 r^2 (1+2\theta'')}. \tag{82}$$

The time τ_g'' within which a body m' colliding with smaller bodies will acquire mass m' can be determined from (73):

$$\frac{1}{\tau_g''} = \frac{\pi r'^2 f_2 \rho'' v}{m'}\left(\frac{m'}{m_1}\right)^{2-q} \approx \frac{4\pi a_0 r'^2 f_2}{Pm'}\frac{v}{\bar{v}}\left(\frac{m'}{m_1}\right)^{2-q}\left(1-\frac{m}{Q}\right). \tag{83}$$

Let us set

$$T_D^* = T_D''/\chi. \tag{84}$$

Then

$$\tau_g = 1\Big/\left(\frac{1}{T_D^*}+\frac{1}{T_D''}\right) = T_D''/(1+\chi). \tag{84'}$$

As for (74), from (76) and (83) we obtain

$$\frac{\tau_z}{\tau_g} \approx \frac{\tau_e''(1+\chi)}{T_D''} = \frac{2-q}{3-q}\frac{8f_1 v}{f_2 v}(1+\chi)\,\theta''^2\left(\frac{m_1}{m}\right)^{4/3}\left(\frac{m_1}{m'}\right)^{2/3-q} = b. \tag{85}$$

Consequently,

$$\theta''^2\left(\frac{m_1}{m}\right)^{4/3} = \theta_1^2 = \frac{3-q}{2-q}\frac{bf_4}{8f_1(1+\chi)}\left(\frac{m_1}{m'}\right)^{q-2/3}, \tag{86}$$

where

$$f_4 = f_2\sqrt{\theta''/\theta''}.$$

The additional factor $(m_1/m)^{4/3}$ gives us a substantially larger value of θ''. The velocities of the bodies turn out to be nearly the same as if there were no embryo m, and the main body governing the velocities, according to (7.12), should be the body m_1. Therefore if we set $v = \sqrt{Gm_1/\theta_1 r_1}$, expression (86) will yield values of θ_1 comparable with the values of θ obtained earlier.

The relative velocity of a body increases because it draws closer to other bodies at various points along its elliptical orbit (various φ, R, v). At the initial stage these conditions of encounter are fulfilled automatically, since the bodies are sufficiently numerous inside the zone. At the final stage they will obviously be fulfilled as long as $T_D'' \leqslant T_D^*$, i. e., as long as m is sufficiently small. The relative velocities are then given by (86).

The encounters of the bodies with the planet embryo m traveling along a nearly circular orbit take place in practice at a fixed distance from the Sun and, therefore, for a fixed value of the relative velocity v. Encounter of a body with the embryo will be followed by a change in v only if its orbit is altered by other bodies. Therefore if there is only one large embryo m inside the planetary zone and for this embryo $T_D^* < T_D''$, the effective relaxation time T_D^*, in encounters between the bodies and embryo will be equal to the relaxation time T_D''. Consequently in this case, in expression (86) one should take $\chi = 1$ for sufficiently large m.

There is reason to believe, however (see Section 26), that the planetary zone originally contained several embryos. As the embryo masses grew their source zones aggregated, the number of embryos decreased, and finally the largest of these became the planet. The intervals $R_{i+1} - R_i$ between adjoining embryos, amounting to several times the radius r_{L_1} of the largest closed Hill region, were smaller than the width $2\Delta R$ of the region within which the bodies moved (see (9.9)), and therefore 2 to 3 embryos could have coexisted inside it simultaneously. To evaluate τ_g under these conditions we will assume that, in addition to the "main" embryo m_0, the planetary zone contained n_1 embryos of mass m/n_2. The relaxation time T_{D1}^* associated with the action of these additional embryos on the body is given by

$$T_{D1}^* \approx \frac{n_2^2}{n_1\left(1 + \frac{2 \ln n_2}{f_3}\right)} T_D^*. \tag{87}$$

For the case $T_D^* < T_{D1}^* < T_D''$ one can take $\tau_g \approx T_{D1}^*/2$. Then instead of (86) we obtain

$$\theta''^2 \approx \frac{n_2^2}{2n_1\left(1 + \frac{2 \ln n_2}{f_3}\right)} \frac{5bf_4}{2f_3}\left(\frac{Q}{m} - 1\right)\left(\frac{m_1}{m'}\right)^{q-5/3}. \tag{88}$$

The number n_1 of embryos can be estimated if we assume that their mass amounted to the fraction α of the total mass $Q - m$ of material not contained in m. Then

$$n_1 = \frac{\alpha (Q - m) n_2}{m}, \tag{89}$$

and from (88) and (75) we obtain

$$\theta''^2 \approx \frac{5bf_4}{2f_1}\left(\frac{m_1}{m'}\right)^{q-5/3} \frac{n_2}{2\alpha\left(1 + \frac{2 \ln n_2}{f_3}\right)} = \frac{10(2-q) n_2 f_4}{(3-q)\alpha(1 + 2\ln n_2/f_3) f_2}\left(\frac{m_1}{m}\right)^{q-5/3} \theta'^2. \tag{90}$$

The values of θ'' turn out to be several times larger than the corresponding values of θ' according to (75). For $n_2 = 5$, $\alpha = 0.5$ and $m_1 = 10^{-3} m$, the value of θ'' is approximately 4 and 2 times larger than θ' if $p = 3.5$, respectively. For $r \sim 3 \cdot 10^8$ cm the values of θ'' lie between 3 and 7 for $p = 3$ and between 4 and 8 for $p = 3.5$.

In conclusion we note that the foregoing estimates of θ were based exclusively on the increase in relative velocities of bodies within the framework of a two-body problem, where encounters are characterized by rotation

of the relative velocity vector without change in its absolute value. No account is taken here of the role of multiple encounters of a body with the planet, which can involve systematic changes in the orbital parameters. There is a known tendency, for example, for planets and satellites to "enter into resonance," leading to the establishment of commensurable periods of revolution among adjacent bodies. The smaller body will experience "induced eccentricity" of orbit, even when the orbit of the larger body is strictly circular (Goldreich, 1965b).

It is therefore not excluded that the actual values of θ were somewhat smaller than the values obtained above.

Chapter 8

STUDY OF THE PROCESS OF ACCUMULATION OF PROTOPLANETARY BODIES BY THE METHODS OF COAGULATION THEORY

24. Solution of the coagulation equation for a coagulation coefficient proportional to the sum of the masses of the colliding bodies

The size distribution function of bodies is one of the most important characteristics of the protoplanetary cluster. On it depended, to a large extent, the relative velocities of the bodies, the extent of their fragmentation in collisions, the rate of growth of planetary embryos, the transparency of the cluster, and the formation of satellite clusters. The geophysical consequences of the accumulation of the Earth also depended largely on the sizes of the bodies that formed the Earth. This applies first and foremost to the initial temperature of the Earth and to the primordial inhomogeneities of its mantle. Studying the size distribution function for bodies in the process of planetary formation is therefore to be regarded as one of the primary tasks of planetary cosmogony.

The process of aggregation of protoplanetary bodies is similar in some respects to the process of coagulation studied in colloidal chemistry, and also to the process of growth of rain droplets studied in meteorology. Thus it is natural to adopt the methods of coagulation theory for its investigation. Unfortunately, in view of the uniqueness of the accumulation process, no single concrete solution of a problem in coagulation theory can be used to describe it. One can only exploit the most general relations of the theory (i. e., essentially, the method) to construct concrete equations and attempt to solve them.

Coagulation theory is concerned for the most part merely with the fusion of particles. In the accumulation process, on the other hand, disintegration (fragmentation) of colliding particles is also important. Accounting for fragmentation adds very considerably to the complexity of the investigation. It is therefore expedient to begin with the simpler instance of accumulation of bodies that have not experienced fragmentation.

In coagulation theory, the foundations of which were laid by Smoluchowski (1936), the equation of chemical kinetics is usually written in the "discrete" form (see, for example, Chandrasekhar, 1943)

$$\frac{d\nu_k}{dt} = 4\pi\left(\frac{1}{2}\sum_{i+j=k}\nu_i\nu_j A_{ij} - \nu_k\sum_{j=1}^{\infty}\nu_j A_{kj}\right), \qquad (1)$$

where ν_k is the number of particles in an element of volume of dimension k, i. e., consisting of k elementary initial particles, and A_{ij} can be termed the coagulation coefficient. This system of equations was solved by Smoluchowski for the simplest case of $A_{kj}=A_0=$ const for a monodisperse initial state.

There also exists an integral form of the coagulation equation (see, for instance, Schumann, 1940; Todes, 1949):

$$\frac{\partial n(m,t)}{\partial t}=\frac{1}{2}\int_0^m A(m',\ m-m')\,n(m',\ t)\,n(m-m',\ t)\,dm' - \\ - n(m,t)\int_0^\infty A(m,m')\,n(m',t)\,dm'; \qquad (2)$$

unlike ν_i, $n(m,\ t)$ is a continuous function of the particle mass m representing the number of particles of mass m (more precisely, within the unit mass interval $\Delta m=1$) in one cm^3; $A(m,\ m')$ is the collision and aggregation probability for particles m and m' (coagulation coefficient). The first term on the right in (1) and (2) represents the number of particles of mass m (with the index k) formed per cm^3 per sec as a result of the aggregation of particles of mass m' and $m-m'$ (i and $j=k-i$). The second term represents the number of particles of mass m combining per cm^3 per sec with other particles, acquiring a different mass as a result.

A different type of equation has been used in astronomy to study the growth of interstellar particles (Oort and van de Hulst, 1946). To facilitate comparison with (2), we will convert it from an equation for $n(r,\ t)$ into an equation for $n(m,\ t)$. It then becomes, in the absence of fragmentations,

$$\frac{\partial n(m,\ t)}{\partial t}=-\frac{\partial}{\partial m}\left[n(m,\ t)\int_0^m A(m,\ m')\,m'n(m',\ t)\,dm'\right] - \\ - n(m,\ t)\int_m^\infty A(m,\ m')\,n(m',\ t)\,dm'. \qquad (3)$$

The integral in the first term on the right is equal to dm/dt, and the first term as a whole represents the changes in $n(m,\ t)$ which stem from the fact that the masses m of all bodies increase due to absorption of all bodies smaller than m during collisions. More briefly, without the last term equation (3) becomes a one-dimensional continuity equation in which $n(m,\ t)$ represents density and dm/dt the analog of velocity. The second term on the right represents the variation in the number of bodies of mass m due to the fact that such bodies fall on larger bodies. An equation such as (3) is also used by Piotrowsky (1953) and Dohnanyi (1967) to study the process of asteroid disintegration. In meteorology Telford (1955) solved a similar equation without the last term on the right.

The fundamental difference between equations (2) and (3) is that, according to the former, different bodies of identical mass m will have different fates depending on what other bodies they collide with. This results in a rapid increase in the size dispersion of bodies in the system, as well as in more rapid growth of the few bodies that accidentally experience more frequent collisions. By contrast, according to equation (3) all bodies of mass m will grow in the same "mean" fashion due to the settling of smaller bodies. This is an accurate description of the growth of the basic mass of

bodies, but certain important laws of random stochastic processes are overlooked in such averaging. Equation (2) is therefore to be preferred.

The assumption that $A(m, m')$ is constant is entirely inapplicable to our case. For $A(m, m') \neq$ const the problem becomes very complex and equation (2) is usually solved by approximate and numerical methods (e.g., Pshenai-Severin (1954), Das (1955)). But such methods can describe only the early stages of the process and do not permit us to follow the growth of bodies over vast time spans, corresponding in our case to a mass increase by 8–10 orders of magnitude. It is therefore desirable to obtain an exact analytic solution, even if it means taking only a qualitatively reasonable expression for the coagulation coefficient.

When allowance is made for the gravitation of the bodies, the coagulation coefficient can be written as

$$A(m, m') = w(m, m')\pi(r + r')^2\left[1 + \frac{2G(m + m')}{V^2(r + r')}\right]V, \tag{4}$$

where $w(m, m')$ is the probability that colliding bodies will combine, r and r' are the radii of bodies of mass m and m' respectively, and V is the velocity of the body m relative to bodies m' before encounter. In the absence of fragmentation one can take $w(m, m') \approx 1$. The relative velocities of the bodies will depend on their masses, though not strongly (see the chapter on velocity dispersion). Quantity $A(m, m')$ depends mainly on the mass and radius of the bodies, which vary over a wide range. For small bodies the collision frequency and $A(m, m')$ are determined by their geometrical cross-section $\pi(r+r')^2$ and are approximately proportional to $m^{1/3}$. For large bodies the cross-section increases due to gravitation (the second term in square brackets predominating) and $A(m, m')$ is approximately proportional to $(m+m')(r+r')$, i.e., $\sim m^{1/3}$ for $m' \ll m$.

The author (Safronov, 1962a) and Golovin (1963) have obtained an analytic solution of (2) for a coagulation coefficient proportional to the sum of the masses of the colliding bodies,

$$A(m, m') = A_1(m + m'), \tag{5}$$

where $A_1 =$ const. This expression for $A(m, m')$ is a kind of "average" between the above expressions for small and large bodies. It gives a qualitatively accurate general pattern of the mass dependence of $A(m, m')$. For this value of $A(m, m')$ equation (2) becomes

$$\frac{\partial n(m, t)}{A_1 \partial t} = \frac{m}{2}\int_0^m n(m', t)\, n(m - m', t)\, dm' - n(m, t)\int_0^\infty (m + m')\, n(m', t)\, dm'. \tag{6}$$

This equation can be solved with the help of the Laplace integral transform. We begin by dropping the second term on the right. Integrating the left and right hand sides of (6) with respect to m, we obtain an expression for the total number of bodies in the system:

$$\frac{\partial N(t)}{A_1 \partial t} = \int_0^\infty \frac{m}{2}\, dm \int_0^m n(m', t)\, n(m - m', t)\, dm' - \int_0^\infty n(m)\, dm \int_0^\infty (m + m')\, n(m', t)\, dm'.$$

The second integral on the right is equal to $2N\rho$. Changing the order of integration, the first integral becomes

$$\frac{1}{2}\int_0^\infty n(m', t)\,dm' \int_{m'}^\infty n(m-m', t)\,m\,dm =$$
$$= \frac{1}{2}\int_0^\infty n(m', t)\,dm' \int_0^\infty n(m, t)(m+m')\,dm = N\rho.$$

Therefore

$$\frac{\partial N(t)}{A_1 \partial t} = -N\rho \text{ and } N = N_0 e^{-A_1\rho t}, \tag{7}$$

where

$$N_0 = \int_0^\infty n(m, 0)\,dm, \quad \rho = \int_0^\infty mn(m, t)\,dm = \text{const}. \tag{8}$$

We replace the required distribution function $n(m, t)$ by the new variable $g(m, t)$ defined by

$$n(m, t) = e^{Nm/\rho - A_1\rho t} g(m, t). \tag{9}$$

Equation (6) then becomes

$$\frac{N_0}{A_1 N}\frac{\partial g(m, t)}{\partial t} = \frac{m}{2}\int_0^m g(m-m', t)\,g(m', t)\,dm'. \tag{10}$$

We now replace the time t by a new independent variable τ:

$$d\tau = \rho A_1 \frac{N}{N_0}\,dt, \quad \tau = 1 - e^{-A_1\rho t} = 1 - \frac{N}{N_0}. \tag{11}$$

When t varies from 0 to ∞, the variable τ varies from 0 to 1. Instead of (10) we obtain

$$\frac{\partial g(m, \tau)}{\partial \tau} = \frac{m}{2\rho}\int_0^m g(m-m', \tau)\,g(m', \tau)\,dm', \tag{12}$$

where the same symbol $g(m, \tau)$ denotes the new function which emerges when t is replaced by τ in $g(m, t)$ with the aid of expression (11). The Laplace transform is applicable to equation (12). From the inverse function $g(m, \tau)$ we pass to its representation $G(p, \tau)$:

$$G(p, \tau) = \int_0^\infty e^{-pm} g(m, \tau)\,dm. \tag{13}$$

Multiplying (12) by e^{-pm} and integrating with respect to m, we find

$$\rho\frac{\partial G}{\partial\tau}=\frac{1}{2}\int_0^\infty e^{-pm}m\,dm\int_0^m g(m',\tau)\,g(m-m',\tau)\,dm'=$$
$$=\frac{1}{2}\int_0^\infty e^{-pm'}g(m',\tau)\,dm'\int_{m'}^\infty e^{-p(m-m')}g(m-m',\tau)\,m\,dm=$$
$$=\frac{1}{2}\int_0^\infty e^{-pm'}g(m',\tau)\,dm'\int_0^\infty (m+m')\,e^{-pm}g(m,\tau)\,dm=-G\frac{\partial G}{\partial p}.$$

For the representation $G(p,\tau)$ we thus obtain a quasilinear partial differential equation

$$\rho\frac{\partial G}{\partial\tau}+G\frac{\partial G}{\partial p}=0. \tag{14}$$

The general solution of this equation has the form

$$G(p,\tau)=G_0[\rho p-G(p,\tau)\tau], \tag{15}$$

where the arbitrary function $G_0(x)$ is determined from the initial data. According to (13), (9) and (15),

$$G(p,0)=\int_0^\infty e^{-pm}g(m,0)\,dm=\int_0^\infty e^{-\left(p+\frac{N_0}{\rho}\right)m}n(m,0)\,dm=G_0(\rho p). \tag{16}$$

From (15) and (16),

$$G(p,\tau)=\int_0^\infty e^{-[\rho p-\tau G(p,\tau)+N_0]\frac{m}{\rho}}n(m,0)\,dm. \tag{17}$$

From here $G(p,\tau)$ can be determined explicitly only in a few instances. If, for example, we take the initial distribution to be

$$n(m,0)=am^{-q}e^{-bm}, \tag{18}$$

$G(p,\tau)$ will then be given by the expression

$$G(p,\tau)\left(p+\frac{2-q}{1-q}b-\frac{\tau G(p,\tau)}{\rho}\right)^{1-q}=a\Gamma(1-q), \tag{19}$$

where $q<1$. For $q=1/2$ and $q=-1$ one obtains a cubic equation for $G(p,\tau)$ which can be solved. For other values of q one obtains algebraic equations of higher order for $G(p,\tau)$. We will confine ourselves to the simplest case where $q=0$. Then

$$n(m,0)=ae^{-bm}, \tag{20}$$

$$a=N_0^2/\rho=N_0/\bar{m}_0,\qquad b=N_0/\rho=\frac{1}{\bar{m}_0}, \tag{21}$$

where $\bar{m}_0$ is the mean body mass at the initial instant. Introducing $n(m,0)$

from (20) into (17) and integrating, we obtain

$$G(p, \tau) = \frac{a}{p + 2b - \rho^{-1}\tau G(p, \tau)}, \tag{22}$$

whence

$$G(p, \tau) = \frac{\rho}{2\tau}\left[p + 2b \pm \sqrt{(p + 2b)^2 - 4b^2\tau}\right]. \tag{23}$$

A characteristic property of the quasilinear equation (9) is the indeterminacy of its solutions (I. G. Petrovskii, 1953). This is evident in the solution (23) from the two signs preceding the square root. The branch point occurs at $p = -2b(1 - \sqrt{\tau})$. It shifts from $p = -2b$ for $\tau = 0$ to $p = 0$ for $\tau = 1$, i. e., for $t = \infty$. A definite single-valued solution can be obtained only in the case where the branch point constantly lies beyond the region of values of p considered here. On the other hand, the function $G(p, \tau)$ obtained by means of the Laplace transform is defined in the complex half-plane $\operatorname{Re} p > s_0$, where s_0 is the growth exponent of the inverse function $g(m, \tau)$. From solution (26) obtained below for $g(m, \tau)$ it is seen that $s_0 = -2b$ when $\tau = 0$ and $s_0 = 0$ when $\tau = 1$. Thus the two restrictions on p coincide at the ends of the interval of variation of τ. By taking $p > 0$, we can simultaneously satisfy, for all values of τ, both the condition imposed on the inverse transform and the condition that there be no branch points in the domain of the variable under consideration. The latter is made possible only by the fact that the quantity τ in equation (14) varies in a bounded interval: as the time increases indefinitely, $\tau \to 1$. This may seem to be accidental from a formally mathematical standpoint. But from the standpoint of physics this result is natural. The equation in question describes a definite physical process, and indeterminacy of the solution would indicate instability and the presence of special points in the process.

We note that a similar result will be obtained for other initial distributions $n(m, 0)$. It is easy to show, for example, that if $n(m, 0)$ is of the form (18) or a δ-function, then for real p, points on the envelope of characteristics will lie within the region of negative p for all $\tau < 1$. In the case of complex p, for sufficiently large $\operatorname{Re} p$ the real and imaginary parts of $G(p, \tau)$, as is seen from (17), will be small on the correctly chosen branch. But when the imaginary part is small the solution for $G(p, \tau)$ differs only slightly from the solution for real p.

Consequently, even for complex p the lower boundary of $\operatorname{Re} p$ can be chosen so that the branch points of $G(p, \tau)$ always lie outside the region under consideration.

Of the two branches of solution (23), only one is suitable (the one with the negative sign in front of the square root), since only then will $G(p, \tau)$ remain bounded for $\tau \to 0$ and tend to zero for $p \to \infty$, as should be the case according to (13). Consequently, the required solution of equation (14) for the initial distribution (20) should be of the form

$$G(p, \tau) = \frac{\rho}{2\tau}\left[p + 2b - \sqrt{(p + 2b)^2 - 4b^2\tau}\right]. \tag{24}$$

To pass from the representation $G(p, \tau)$ to the inverse function, one must carry out an inverse Laplace transformation. Let us use the transformation

given in the manual of Ditkin and Kuznetsov (1951),

$$p-\sqrt{p^2-a^2} \doteq \frac{a}{m} I_1(am), \tag{25}$$

where $I_1(x)$ is a modified Bessel function. Using the displacement theorem, we obtain the following expression for $g(m, \tau)$:

$$g(m, \tau) = \frac{N_0}{m\sqrt{\tau}} e^{-2bm} I_1(2bm\sqrt{\tau}). \tag{26}$$

Finally, with (9), (11) and (21) we pass to the required distribution function

$$n(m, \tau) = \frac{N_0(1-\tau)}{m\sqrt{\tau}} e^{-(1+\tau)bm} I_1(2bm\sqrt{\tau}). \tag{27}$$

The function $I_1(x)$ has the following expansions:
for $x \ll 1$

$$I_1(x) = \frac{x}{2}\left[1 + \frac{(x/2)^2}{1^2 \cdot 2} + \frac{(x/2)^4}{1^2 \cdot 2^2 \cdot 3} + \cdots\right], \tag{28}$$

and for $x \gg 1$

$$I_1(x) = \frac{e^x}{\sqrt{2\pi x}}\left(1 - \frac{3}{8x} - \frac{15}{128x^2} - \cdots\right). \tag{29}$$

Correspondingly, the mass distribution function of the bodies (27) can be approximated as follows:
for $2m\sqrt{\tau} \ll \bar{m}_0 = 1/b$

$$n(m, \tau) \approx N_0 b(1-\tau) e^{-(1+\tau)bm}, \tag{30}$$

and for $2m\sqrt{\tau} \gg \bar{m}_0$

$$n(m, \tau) \approx \frac{N_0(1-\tau)}{2\sqrt{b\pi}\tau^{3/4}} m^{-3/2} e^{-(1-\sqrt{\tau})^2 bm}. \tag{31}$$

It is only at the early stage of the process of aggregation and for small values of m that the condition that $2m\sqrt{\tau} \ll \bar{m}_0$ is met and that the distribution function is exponential. For most of the region of values of m and τ, one can use expression (31), which is the product of a power of m by an exponential function. For large t the value of τ is close to 1 and $(1-\sqrt{\tau})^2$ is very small. In this case, therefore, the exponential function will begin to play an important role only for the largest m. The condition $n(m) \propto m^{-3/2}$ is met over nearly all the interval of variation of m (except for the largest and smallest m). We note that this power function is close to the mass distribution obtained observationally for small bodies in the solar system — comets, asteroids, meteorites reaching the Earth. The exponent $-3/2$ of m is independent of the parameters a and b of the initial distribution and is apparently determined only by the form of the coagulation coefficient $A(m, m')$.

In distribution (31), a considerable portion of the mass of the system consists of large bodies; with time, moreover, the relative mass of the

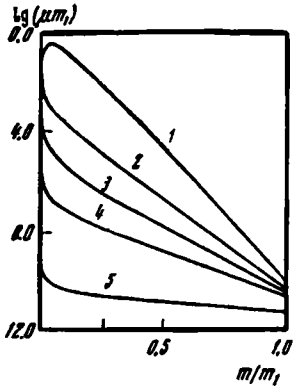

FIGURE 4. Mass distribution function $\mu(m)\Delta m = mn(m)\, m_1$ for bodies, obtained from the analytic solution of the equation for coagulation without fragmentation. Quantity m_1 is the mass of the largest body in the distribution and is taken as unity along the horizontal axis:

$1 - N = N_0$ $(t = 0)$; $2 - N = 10^{-1} N_0$; $3 - N = 10^{-3} N_0$; $4 - N = 10^{-5} N_0$; $5 - N = 10^{-6} N_0$.

large bodies increases. Figure 4 indicates the mass distribution $\mu(m) = mn(m)$ at different instants corresponding to reduction by factors of 10, 10^2, 10^3 and 10^5 in the total number N of bodies in the system compared with the initial number N_0. The unit of measurement of m is taken to be the mass of the "largest body" m_1, obtained by integrating the "tail" of the mass distribution function, which contains one body:

$$m_1 = \int_{M_1}^{\infty} mn(m)\, dm,$$

where

$$\int_{M_1}^{\infty} n(m)\, dm = 1.$$

It would of course be rash to apply this result directly to the process of planetary accumulation, as no account has been taken here of the fragmentation of colliding bodies, which increases the amount of fine substance in the system. But the result that large bodies played a considerable part in the accumulation process seems to be correct.

25. Asymptotic power solutions of the coagulation equation

Owing to the complexity of the coagulation equation, especially when fragmentation is allowed for, there is very little hope that an analytic solution for it will be found. Hence the importance of qualitative methods of investigating the equation that shed light on the nature of the distribution function without seeking an actual solution. Such methods include attempts to seek partial solutions that could be regarded as asymptotic. Piotrowsky (1953), for instance, having proposed a power form of solution to his equation for the size distribution of asteroids involving relatively simple analysis, arrived at the conclusion that the radius distribution of asteroids must tend to a

power distribution with exponent $p = 3$. Our equation is considerably more complicated but it too can be submitted to analysis of this kind.

a) **Accumulation in the absence of fragmentation.** Consider equation (6). For greater generality we will take finite limits of integration:

$$\frac{\partial n(m, t)}{A_1 \partial t} = \frac{m}{2} \int_{m_0}^{m} n(m', t)\, n(m - m', t)\, dm' - n(m, t) \int_{m_0}^{M} (m + m')\, n(m', t)\, dm', \tag{32}$$

where m_0 and M are the lower and upper bounds in the distribution. We assume that at a certain instant t the distribution function has the form

$$n(m) = cm^{-q}, \tag{33}$$

where c and q are independent of m. Introducing the value of $n(m)$ in (32) and setting $m'/m = x$, we obtain

$$\frac{\partial n}{A_1 c n \partial t} = m^{2-q} \left[\int_{\frac{m_0}{m}}^{\frac{1}{2}} x^{-q} (1 - x)^{-q}\, dx - \int_{\frac{m_0}{m}}^{\frac{M}{m}} (1 + x)\, x^{-q}\, dx \right] =$$

$$= m^{2-q} \left\{ \int_{\frac{m_0}{m}}^{\frac{1}{2}} x^{-q} [(1 - x)^{-q} - 1]\, dx + \frac{1}{q - 1} \left[\left(\frac{m}{M}\right)^{q-1} - 2^{q-1} \right] \right\} -$$

$$- \frac{1}{2 - q} (M^{2-q} - m_0^{2-q}) = m^{2-q} F\left(q, \frac{m}{M}, \frac{m_0}{m}\right) - F_1(q, M, m_0). \tag{34}$$

Expressing c in terms of the total mass of material per cm^3, which we will regard as constant, we obtain, for $q \neq 2$,

$$\frac{c}{2 - q} (M^{2-q} - m_0^{2-q}) = \rho$$

and

$$\frac{\partial n}{A_1 n \partial t} = \frac{(2 - q)\rho}{1 - (m_0/M)^{2-q}} \left(\frac{m}{M}\right)^{2-q} F\left(q, \frac{m}{M}, \frac{m_0}{m}\right) - \rho. \tag{34'}$$

Relation (34) tells us in what direction the variation of $n(m)$ proceeds. The second term on the right, which is independent of m, leads to the same relative reduction in $n(m)$ for any m, i.e., it characterizes the decrease of c in (33). The first term gives the relative variation in $n(m)$ depending on m. It therefore produces variation in q. If, for instance, it is positive and increases with increasing m, the fraction of large bodies will increase and q will decrease.

In the general case the variation in q is different for different m, the mass distribution of the bodies deviates from the power law immediately, and relations (34) and (34') become inapplicable. But for the initial instant when the distribution by assumption follows a power law, these relations hold true and indicate the direction of variation in q. By introducing various values of q in them one can try to obtain an asymptotic distribution.

In order for a power solution of the coagulation equation with an exponent q to exist, it is necessary that for this value of q the right-hand side of (34) and (34') be independent of m. Of greatest interest, therefore, are the values $q=q_0$ that are roots of the equation $F\left(q, \frac{m}{M}, \frac{m_0}{m}\right)=0$. If the coagulation equation is of a form such that F is independent of m, then the root q_0 will also be independent of m. Then the power distribution (33) with exponent q_0 is a solution of the coagulation equation. This holds, for example, for the coagulation equation under consideration when $m_0 = 0$ and $M=\infty$. For $F(q)$ to converge in this case, it is necessary that $q < 2$ (but $cF(q)$ in (34') will converge even for $q > 2$). Then

$$F(q)=\frac{q}{2^{2-q}}\left[\frac{1}{2-q}+\frac{1+q}{2\cdot 2(3-q)}+\frac{(1+q)(2+q)}{2^2\cdot 2\cdot 3(4-q)}+\cdots\right]-\frac{2^{q-1}}{q-1}=0. \qquad (35)$$

Since the solution to equation (6) corresponding to the initial distribution (20) is already known (see (27) and (31)), our first step must be to check whether the value $q = 3/2$ is a root of equation (35). A simple substitution will convince us that this is so. Consequently, the function $n(m)=cq^{-3/2}$ is truly a solution of (6).

However, not every root q_0 of $F(q)=0$ will give us an asymptotic solution. An asymptotic distribution should be stable, i. e., distributions with q approaching q_0 should tend to it. For $q_0 < 2$ this condition will be met if $F'(q_0) > 0$. Then for $q < q_0$ we will have $F(q) < 0$ and, from (34'), the relative decrease in n will increase with m; therefore q should increase until it reaches q_0. For $q > q_0$ we have $F(q) > 0$ and $\partial n/n\partial t$ increases with increasing m. The relative fraction of large bodies increases and q decreases until it reaches q_0. Consequently, $q \to q_0$ from both sides and the distribution cm^{-q_0} is indeed asymptotic. By contrast, for $F'(q_0) < 0$ the exponent q_0 gives an unstable solution: for $q < q_0$, q decreases while for $q > q_0$, q increases. Such a solution could not be asymptotic since solutions close to it would diverge from it in the course of time.

For $q_0 > 2$ the solution is asymptotic if $F'(q_0) < 0$. Lastly, $q = 2$ is a special value. It gives an asymptotic solution for $F(q=2) < 0$.

Thus the condition that an asymptotic solution of the power form type should satisfy can be written as:

$$\begin{array}{lll} F(q)=0,\ F'(q)>0 & \text{for} & q<2, \\ F(q)<0 & \text{for} & q=2, \\ F(q)=0,\ F'(q)<0 & \text{for} & q>2. \end{array} \qquad (36)$$

From (35) it is seen that when q decreases from $3/2$ to 1, $F(q)$ decreases from 0 to $-\infty$. Thus $F'(3/2) > 0$. This means that $q_0 = 3/2$ should give an asymptotic solution.

This emerges from the behavior of solution (31). From (11)

$$(1-\sqrt{\tau})^2 bm \approx \frac{1}{4}\left(\frac{N}{N_0}\right)^2 \frac{m}{\bar{m}_0}=\frac{1}{4}\frac{N}{N_0}\frac{m}{\bar{m}}.$$

In the course of time m increases, but then $\bar{m}$ also increases, while N/N_0 decreases. The role of the factor with a power of m in (31) will therefore decrease constantly for the same $m/\bar{m}$. The body distribution will tend to follow a power law over an increasingly large interval of values of m.

The foregoing discussion is largely formal, since the power distribution law is physically inapplicable over the entire infinite interval of variation of m. The assumption $m_0 = 0$ and $M = \infty$ means that either $p = \infty$ or $c = 0$. In the former case the second term F_1 in (34) will diverge, while in the latter for $q < 2$ the first term in (34') will tend to zero. Nevertheless the above method of qualitative analysis of the coagulation equation makes it possible to obtain asymptotic solutions of power form. While the power solution is in itself physically inapplicable, it is the limit of real solutions that do not possess its defects. Thus in solution (31) the principal term is the power $m^{-9/4}$ and there is an additional exponential factor which removes the divergence of the asymptotic power solution. The form of the additional factor is probably determined by the form of the initial distribution $n(m, 0)$, whereas the character of the asymptotic solution reflects the properties of the equation. In the presence of the exponential factor the difference between distribution functions with finite and infinite limits M is insubstantial, since $n(m)$ decreases rapidly and not a single body remains throughout the interval of values of m greater than a certain M: $\int_M^\infty n(m)\,dm < 1$. In practice a distribution with $M = \infty$ is equivalent to a distribution with finite M to which has been added, in the region of largest $m \sim M$, an additional finite mass given by $\int_M^\infty mn(m)\,dm$. But mathematically the difference between these distributions is considerable: for the one there exists an asymptotic power solution, for the other no such solution exists and all solutions are more complex. Since, however, for $m_0 \neq 0$ and $M \neq \infty$ the function $F\left(q, \frac{m}{M}, \frac{m_0}{m}\right)$ is not very different from $F(q)$ in (35), if $m_0 \ll m \ll M$ one might expect that an asymptotic solution for this range of values of m should be close to the asymptotic solution for $m_0 = 0$ and $M = \infty$, i. e., close to a power function with $q_0 = 3/2$.

b) **Allowance for the fragmentation of colliding bodies.** Fragmentation of colliding bodies played an important role in the process of planetary accumulation. By increasing the amount of fine substance in the system, fragmentations exercised considerable influence on the size distribution function of the bodies. Quantitative treatment of this effect is very difficult. Even without allowing for fragmentation, the coagulation equation is too complex for there to be any hope of obtaining an analytic solution. Moreover, the fragmentation process itself has hardly been studied, and no reliable data exist concerning the size distribution of fragments in impacts from large bodies. A well-known logarithmic law of size distribution was obtained by Kolmogorov from purely probabilistic considerations for the case of multiple fragmentations of particles experiencing any kind of disintegration. A more general expression was later found by Filippov (1961).

We note that the probability for disintegration of colliding bodies with a significant gravitational attraction varies for different mass ratios. It is known that a particle striking the surface of a larger body at a velocity of 5—10 km/sec will form a crater, scooping out from it a mass 2—3 orders of magnitude greater than the mass of the particle. Therefore the mean velocity of ejection will be 1—1.5 orders of magnitude less than the velocity of the

impacting particle. If the body is massive enough the ejected matter will be unable to overcome its attraction and will fall back on the body. In this case fragmentation in the above sense (i. e., of disintegration) will not take place. A different picture emerges for the collision of bodies of comparable mass at the same velocity. Here the impact energy per unit mass for both bodies is much greater, and correspondingly the velocity of dispersion of matter will be considerably larger. Consequently, the probability for disintegration (fragmentation) in collisions between bodies of comparable mass is considerably larger than for substantially different masses.

Let $w(m, m')$ be the probability that the bodies m and m' will combine in collision and $1-w(m, m')$ the probability that they will fragment. Further, let $n_1(m, m'')$ be the distribution function for the mass m of the fragments resulting from collisions between two bodies of total mass m''. Obviously, $m < m''$ and

$$\int_0^{m''} n_1(m, m'')\, m dm = m''. \tag{37}$$

Allowance for fragmentation introduces the following change in the coagulation equation (2). In the first integral the integrand is multiplied by $w(m', m-m')$. The second integral remains the same. A third term is introduced, characterizing the increase in the number of bodies per unit time due to fragmentations. Two-body collisions with total mass m'' gives an increment $n_1(m, m'')\, N(m'')$ in the number of bodies m, where $N(m'')$ is equal to the first integral of equation (2), in which the integrand has been multiplied by $1-w(m', m''-m')$ and m'' replaces m. The total increment in the number of bodies m is obtained by integrating this expression over all $m'' > m$. Consequently, the equation has the form

$$\begin{aligned}\frac{\partial n(m, t)}{\partial t} = &\int_0^{m/2} w(m', m-m')\, A(m', m-m')\, n(m', t)\, n(m-m', t)\, dm' - \\ &- n(m, t) \int_0^{\infty} A(m, m')\, n(m', t)\, dm' + \\ &+ \int_m^{\infty} n_1(m, m'') \int_0^{m''/2} [1 - w(m', m''-m')]\, A(m', m''-m') \times \\ &\times n(m', t)\, n(m''-m', t)\, dm' dm''. \end{aligned} \tag{38}$$

A preliminary examination of this equation was carried out by the author under the assumption that

$$w(m', m-m') = 1 \qquad \text{for} \quad m' < \frac{m}{2}(1-\alpha)$$

and

$$w(m', m-m') = 0 \qquad \text{for} \quad m' > \frac{m}{2}(1-\alpha) \tag{39}$$

(total disintegration of bodies of comparable size). The distribution function over m of the fragmented material was taken as

$$n_1(m, m'') = c m^{-1} e^{-b_1 m/m''}, \tag{40}$$

and the coagulation coefficient in the form (5). The body mass $\mu(m, t) = mn_1(m, t)$ was taken as the unknown function. After transformation to remove the divergence of the first two integrals for $m \to 0$, the coagulation equation assumed the form

$$\frac{\partial\mu(m, t)}{A_1\partial t} = m\int_0^m \mu(m', t)\frac{\mu(m-m', t)-\mu(m, t)}{m'}dm' - m\mu(m)\int_m^\infty \frac{\mu(m')\,dm'}{m'} - \rho\mu(m, t) - m\int_{\frac{m}{2}(1-\alpha)}^{\frac{m}{2}1+\alpha} \frac{\mu(m', t)\mu(m-m', t)}{m'}dm' + b_2\int_m^\infty e^{-b_1 m/m''}\int_{\frac{m}{2}(1-\alpha)}^{\frac{m}{2}(1+\alpha)} \frac{\mu(m', t)\mu(m-m', t)}{m'}dm'dm''. \quad (41)$$

Searching for an asymptotic power solution indicated that condition (36) is met only for very small values of α. The root of the equation $F(q, \alpha\ldots) = 0$ which yields an asymptotic solution moves with increasing α from $q_0 = 3/2$ for $\alpha = 0$ toward smaller values of q. For $\alpha > 10^{-2}$ the function $F(q, \alpha)$ has no roots in the region of values of $q < 2$. To check whether this result could have been due to the form of the distribution function adopted for the fragmented bodies (40), a similar calculation was carried out for the following distribution:

$$n_1(m, m'') = c\left(\frac{m}{m''}\right)^{-q_1}. \quad (42)$$

It was found that, in this case as well, condition (36) is met only for small α, and the roots q_0 move with increasing α toward smaller q. Table 10 gives the values of q_0 for $n_1(m, m'') = c\left(\frac{m}{m''}\right)^{-q_1}$ as a function of the parameters α, q_1, and $M' = M/m$.

TABLE 10

q_1	M'	α						
		0	0.01	0.05	0.07	0.10	0.12	0.15
0.4	∞	1.5		1.48	1.47	1.45	1.42	—
0.4	100		1.47	1.44		1.37	—	—
0.7	∞	1.5		1.46	1.40	—	—	—
1.2	100		1.45	—	—	—	—	—
1.5	100		1 43	—	—	—	—	—

For $M' = \infty$ a solution exists only for $q_1 < 1$, while for $M' \neq \infty$ there is no exact power solution, since q_0 is slightly different for different M'. Unfortunately, solutions that hold only for small α cannot describe gross properties of the system associated with fragmentations.

We were able to obtain an asymptotic solution for larger values of the fragmentation coefficient α for a power function $n_1(m, m'')$ of the form (42), for the special case where $q_1 = q$ (Figure 5). Below we list the values of the exponent q_{01} of this power solution as a function of α:

α	0	0.1	0.2	0.4	0.6	0.8
$A = A_1(m + m')$	1.5	1.55	1.62	1.74	1.85	1.94
$A = A_1 m$	—	0.94	1.13	1.25	1.34	1.41

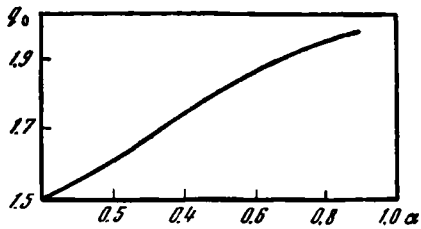

FIGURE 5. Dependence of the exponent q_0 in the asymptotic power solution of the coagulation equation (with fragmentations) on the parameter α for the case where q_1 equals the exponent of the required distribution.

In the third row of the table we list for comparison the values of q_{01} for the same premises but a slightly different coagulation coefficient: $A = A_1 m$, where m is the mass of the larger of two colliding bodies. The differences in q_{01} are easy to understand: as the bodies being fragmented are of nearly the same size, for $A \propto (m+m')$ the fragmentations are nearly twice as intensive as for $A \propto m$, and q_{01} should be larger. The form $A_1 \cdot (m+m')$ is closer to reality. The most likely values of the exponent of the asymptotic power solution are apparently $q_{01} = 1.8$—1.9. The exponent q_{01} remains less than 2 despite the fact that q_1 must necessarily increase with α. The more general case of $q \neq q_1$ has been analyzed by E. V. Zvyagina. She found that q tends to a certain value between q_{01} and q_1 which is closer to q_1 as m decreases. Since the values of q depend on m, there is no exact asymptotic power solution. Still, from the foregoing results one can conjecture that approximation of the size distribution function by a power function is permissible in a system of bodies with fragmentations and that the exponent q of this function will lie between 1.5 and 2 (more likely closer to 2) depending on the intensity and character of the fragmentations. In the large body region ($m \sim M$) the deviations of the distribution functions from the power function will be maximum, but in the region of small bodies ($m \ll M$) the power approximation is entirely satisfactory.

Roughly the same values for q are obtained in the limiting case where there is no accumulation, only fragmentations taking place. An equation of type (3) was studied for asteroid fragmentation by Dohnanyi (1967) as well as by Piotrowsky. However, Dohnanyi assumed that fragmentation produced not only fine particles but also larger fragments with a power mass distribution law of exponent $q_1 \approx 1.8$, as follows from experimental data. For the stationary case $dn/dt = 0$, Dohnanyi obtained a power solution with exponent $q \approx {}^{11}/_{6}$.

It should be mentioned that the foregoing results regarding q agree with factual data on distributions of asteroids (Piotrowsky, 1954; Jashek, 1960), meteorites (Braun, 1960) and comets (Öpik, 1960), which lead to $q \approx 1.6$—1.8.

An attempt was made by the author to calculate the function $n(m, t)$ on a BESM-2 computer for the above premises regarding A, w and n_1, i. e., in the form (5), (39) and (40). Simplifications made in the program did not allow the variation of $n(m, t)$ to be followed over a large time span. Nevertheless,

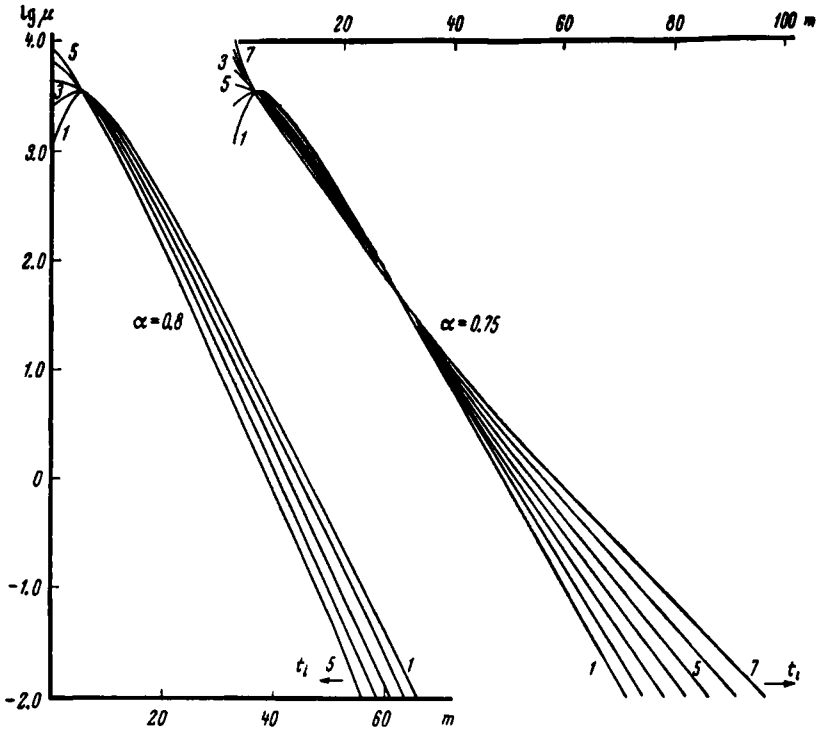

FIGURE 6. Time variation of the mass distribution function for bodies for two nearby values of the parameter α characterizing the fragmentation intensity of colliding bodies (calculations performed on a BESM-2 high-speed computer).

it was found that the variation of the distribution function was substantially different for different values of the fragmentation parameter α. For $\alpha < 0.78$ accumulation predominates and in the course of time the number of larger bodies increases, the entire distribution shifting toward larger m. For $\alpha > 0.78$ the picture is reversed: the number of large bodies decreases and the distribution shifts toward small m. The value $\alpha \approx 0.78$ corresponds to the root q_0 characterizing an unstable solution, according to the terminology introduced earlier (Figure 6).

Thus from preliminary analysis it is already apparent that fragmentation of colliding bodies has a substantial influence on the size distribution function established during the accumulation process. It is important that information on the general character of this function should be found, and therefore data regarding the nature of fragmentations in collisions must be made more precise. It is necessary to determine to what extent the available laboratory data on fragmentations of small particles are applicable to the description of fragmentations of large bodies. In view of the great difficulties that arise in the solution of the coagulation equation, it becomes especially important to develop methods for analyzing it qualitatively and search for asymptotic (and in particular, power) solutions.

Chapter 9

ACCUMULATION OF PLANETS OF THE EARTH GROUP

26. Growth features of the largest bodies

When the distribution function for protoplanetary bodies is studied by the coagulation theory method, certain fundamental laws of the accumulation process fail to emerge. Asymptotic solutions are a good representation of the distribution function in the region of small and medium-sized bodies. As to the growth of the largest bodies, much remains obscure. The concept of a distribution function is applicable only as long as the number of bodies (within a given size interval) is large enough to permit the use of statistics. But statistics cannot be applied to the largest individual bodies. Yet these may be the most interesting of all, since one of them eventually becomes the "embryo" of the future planet. It is therefore fitting to dwell at greater length on their growth pattern.

It is widely held that if two bodies of different size are placed in a medium which supplies them with material, then the masses of the bodies will tend to equalize as they grow. This is valid as long as the rate of mass growth of a body is proportional to its geometric cross section, i. e., to the surface area of the body. For the smaller body the surface area per unit mass is greater, and its relative increment should be greater, than for the larger body. However, these considerations do not apply to the largest bodies. Owing to gravitation their effective cross sections are considerably larger than the geometric cross sections and are proportional to a higher power of mass than the first. In this case the mass difference (ratio of larger to smaller) will increase and not decrease with time.

If a body (m, r) is large compared with other bodies m', then $m+m'\approx m$, $r+r'\approx r$, and its collision cross section, after allowing for gravitation, can be written as

$$\pi l^2 \approx \pi r^2\left(1+\frac{2Gm}{V^2 r}\right), \tag{1}$$

where V is the velocity of incident bodies relative to m before approach to m. The velocities of very large bodies are usually small, and V is, in practice, the mean velocity v of incident bodies relative to the circular velocity. Expressing it according to (7.12) in terms of the mass and radius of the largest body in the form $v^2=Gm/\theta r$, we obtain

$$l^2 \approx r^2(1+2\theta). \tag{2}$$

Since θ is of the order of a few units, the second term dominates and

$$l^2 \approx 2\theta r^2 \approx \frac{2Gmr}{v^2} \propto r^4/v^2. \tag{3}$$

For the body next in size (m_1) situated in the same zone, we obtain

$$l_1^2 \approx r_1^2\left(1 + \frac{2Gm_1}{v^2 r_1}\right) = r_1^2\left(1 + 2\theta\frac{r_1^2}{r^2}\right). \tag{4}$$

As long as m_1 is comparable with m, the second term on the right in (4) will predominate and therefore

$$l^2/l_1^2 \approx r^4/r_1^4.$$

The ratio of the effective cross sections of the largest bodies is proportional to the fourth power of the ratio of their radii. Therefore

$$\frac{dm/m}{dm_1/m_1} \approx \frac{r}{r_1} > 1. \tag{5}$$

It is easy to show that the largest body m will then grow more rapidly than the body m_1 both absolutely and relatively, i. e., the ratio m/m_1 increases with time.

Consequently, the largest body outstrips the others as it grows, pulling apart from them, as it were. However, the difference between the masses m and m_1 can increase only to a certain limit. When m_1 becomes much smaller than m, the second term in (4) becomes smaller than the first and the collision cross section of m_1 approaches the geometric cross section:

$$l_1^2 \approx r_1^2. \tag{4'}$$

Then $l^2/l_1^2 \approx 2\theta r^2/r_1^2$ and

$$\frac{dm/m}{dm_1/m_1} \approx 2\theta\frac{r_1}{r}. \tag{6}$$

The ratio m/m_1 stops increasing when the right side of (6) decreases to unity, i. e., when

$$r \approx 2\theta r_1, \quad m \approx (2\theta)^3 m_1. \tag{7}$$

The values $\theta \approx 3-5$ obtained in Chapter 7 give

$$m \approx (0.2 - 1)\cdot 10^3 m_1.$$

Thus a characteristic feature of the accumulation process was the formation of larger bodies or planet "embryos" whose masses were far greater than the masses of the other bodies. In Chapter 11 it will be shown that the inclinations of the axes of rotation of the planets are due to the randomly oriented impacts of large bodies falling on the planets. It has been possible to estimate the masses of the largest of these from the degree of axial inclination (Table 12). For the Earth it was found that $m/m_1 \approx 10^3$. Expression (7) then yields the value $\theta = 5$, which agrees well with the data for θ in

Table 7. Since estimates for m/m_1 based on axial inclination are fairly reliable, (7) can be used for an independent evaluation of θ. We recall that in deriving (7) we adopted the simplified relation (1): we assumed that $r+r' \approx r$ and $m+m' \approx m$. More detailed treatment leads to a cumbersome expression instead of (7) from which the above value of m/m_1 is obtained for a somewhat larger θ.

The limiting ratio m_1/m could only have been actually reached for bodies m_1 that had remained over long periods within the zone of the largest body m, which controlled the relative velocities v of the bodies according to (7.12). As m/m_1 increased the orbit of m became more and more near-circular and the supply zone of m comprised an annular region of width $2\Delta R$ determined by the mean orbital eccentricity e of the main mass of bodies. If R is the orbital radius of m, then

$$\Delta R = eR \approx \frac{\sqrt{2}\,v}{\omega} = \frac{1}{\omega}\sqrt{\frac{8\pi G\delta}{3\theta}}\,r. \tag{8}$$

All bodies with semimajor axes lying within the interval $R \pm \Delta R$ could have collided with m. The distance to the libration point L_1 of the body m,

$$r_{L_1} = \left(\frac{m}{3M}\right)^{1/3} R = \left(\frac{4\pi G\delta}{9\omega^2}\right)^{1/3} r \tag{9}$$

is more than one order smaller than ΔR. In practice, therefore, the zone of gravitational influence of m did not extend beyond the supply zone. Such a body m traveling along a nearly circular orbit and having a mass substantially greater than that of other bodies inside the zone can be called a planetary "embryo." As long as the bodies remained small, velocities and eccentricities were small and the zones $2\Delta R$ were narrow. Outside the zone under consideration lay others in which velocities were determined by the largest bodies inside them, which, in turn, grew relatively more rapidly than other bodies, their orbits tending to become circular. Initially, therefore, there were many planet "embryos." As long as they occupied different zones they were not affected by the law that the largest of all should grow relatively more rapidly, but due to random factors their masses could have varied by several times. As the bodies grew, so did the velocities, and the zones broadened. Where adjacent zones overlapped, velocities equalized and the smaller embryo began to grow more slowly; but it continued along a nearly circular orbit for a long time, as the distance $\Delta R'$ between embryo orbits for which the smaller of the embryos is susceptible to strong perturbations from the larger is only three to five times greater than the r_{L_1} of the embryo m and is several times smaller than ΔR. Fusion of the embryos took place only after m had increased by 1.5 – 2 orders of magnitude. Within this time the mass ratio must have increased considerably. The gradual reduction in the embryo population due to mass increase continued until all the surrounding material had been used up and the distances between embryos had become so large that mutual gravitational perturbations were unable to disrupt the stability of their orbits over large time spans. This is the principal condition governing the law of planetary distances.

In the last stage of growth the process of accumulation was significantly more complex, since the increase in relative velocities caused fragmentations in collisions to play an important role. Collision between embryos and

even the largest bodies m_1 would not lead to their disintegration. When m_1 falls on m, the kinetic energy of the impact per unit mass of m_1

$$\frac{v_1^2}{2}+\frac{Gm}{r}=\frac{Gm}{r}\left(1+\frac{1}{2\theta}\right) \tag{10}$$

is only one tenth greater than the potential energy at the surface Gm/r. During impact a considerable fraction of the kinetic energy changes into heat. The remaining energy is expended in ejecting from the area a crater of mass far in excess of m_1. Therefore the kinetic energy of dispersion per unit mass is on the average far less than the potential energy at the surface of m, and the ejected material cannot leave the embryo m.

For other bodies collisions are more dangerous. The impact energy of the body m' when it falls on m_1 is given by

$$m'\left(\frac{Gm}{2\theta r}+\frac{Gm_1}{r_1}\right)=m'\frac{Gm_1}{r_1}\left(\frac{r^2}{2\theta r_1^2}+1\right)=(2\theta+1)m'\frac{Gm_1}{r_1}. \tag{11}$$

For m' close to m_1 the impact energy is considerably larger than the total potential energy of the colliding bodies. In this case collision will lead not to fusion but rather to destruction of both bodies, to their disintegration into many fragments. If one half of the impact energy changes into mechanical energy of dispersion, then for $\theta = 5$ disintegration will occur when $m' > 0.15\,m_1$; if one third of the impact energy so changes, disintegration will occur for $m' > 0.25\,m_1$. Bodies m_1 could have grown only by collision with bodies of considerably smaller mass. When allowance is made for fragmentation, therefore, the limiting ratio m/m_1 should be greater than the value given by expression (7). At first a smaller embryo entering the zone of a larger embryo m will merely lag behind it in growth and will not disintegrate in collisions with other bodies. But as the growth lag widens collisions become increasingly perilous and it may be destroyed before it collides with the largest embryo m.

Using the limiting theorems of probability theory and assuming an inverse power law of mass distribution with exponent $q < 2$, Marcus (1967) calculated the mathematical expectation of the masses of the largest bodies and concluded that there was no marked gap between the masses of the largest body and those of the next largest ones. He obtained $m/m_1 \approx 3$ for $q = 3/2$. Marcus seeks to circumvent the difficulties with planetary rotation that arise here by supposing that the bodies fell on the planet with a velocity far less than the parabolic velocity at the surface of the planet. The mathematical expectation of the mass ratios of the largest bodies m_k/m_{k+1} is also easy to find without the cumbersome apparatus of probability theory, that is, directly from the size distribution function used for the bodies (Safronov and Zvjagina, 1969). The ratios obtained by Marcus will then emerge for the particular case where the mass of the largest body $m=2-q$. The power law of distribution, however, does not take into account the growth characteristics of the largest bodies (considered above) and gives a poor description of their size distribution. The assumption that impact velocities were small is strange, to say the least, as there were no forces capable of significantly slowing the motion of bodies inside the gravitational field of the planet.

27. Accumulation of planets of the Earth group

From the foregoing it emerges that, despite the complexity of the accumulation process and the fact that fragmentation among colliding bodies was important, the process of growth of the largest bodies (the planetary "embryos") can be described quantitatively in an entirely satisfactory manner if we assume that their growth resulted from the settling on them of significantly smaller bodies and that they were not fragmented during these collisions. We can also assume that they moved at all times along circular orbits $\rho=\rho_0$ situated in the central plane of the cluster where the density of matter is maximum. The function $\rho(z)$ inside the cluster can be taken in the form of the barometric formula (3.12) derived for gases.

The mass increment which the embryo acquires when it uses up other bodies can be written in the ordinary form

$$\frac{dm}{dt}=\pi r_e^2\rho_0 v, \tag{12}$$

where πr_e^2 is the effective collision cross section and v the mean velocity of the bodies relative to the embryo, i.e., in practice the velocity relative to the circular Kepler velocity at the given distance from the Sun. For bodies with gravitational interaction $r_e=l$, where l is given by (1) and (2).

As relative velocities v increase, so does the uniform thickness H of the cluster, and the density ρ_0 decreases. According to (3.5) the product $\rho_0 v$ is independent of v here and is determined only by the surface density σ (Safronov, 1954).

In Chapter 7 we saw that while the mean relative velocities of bodies of different masses are not the same (different θ), these differences are small and it is possible to speak of a mean velocity of the entire set of bodies. The surface density σ of matter drops owing to the exhaustion of material by the planetary embryo m. For a closed planetary zone that does not exchange material with other zones, one can write

$$\sigma=\sigma_0\left(1-\frac{m}{Q}\right), \tag{13}$$

where σ_0 is the complete (initial) surface density, including the embryo m, and Q is the present mass of the planet.

The expression (12) for the rate of growth of the planetary embryo can now be written as

$$\frac{dm}{dt}=\frac{4\pi(1+2\theta)}{P}\sigma_0\left(1-\frac{m}{Q}\right)r^2. \tag{14}$$

This expression differs from Shmidt's (1945) well-known formula for the rate of planetary growth by a factor $2(1+2\theta)$. The factor $(1+2\theta)$ comes from the increase in effective collision cross section compared with the geometric cross section due to gravitational focusing. The numerical factor 2 stems from the fact that in Shmidt's formula, which was derived by different means, when evaluating the collision frequency only motions along the z direction were taken into account. It was assumed that within the time P of revolution around the Sun, any body will intersect the orbital plane of the planet twice, the probability of its falling on the planet being

equal both times to the ratio of the planet's cross section πr^2 to the area of the planetary zone Q/σ_0.

From (14), on the basis of (7.82), we obtain the obvious relation

$$\frac{dm}{dt}=\frac{Q-m}{\tau_e^*}, \tag{15}$$

where τ_e^* is the expectation value of the time preceding collision with the growing planet for a body traveling randomly in its zone. It is essentially the characteristic time of exhaustion of the planetary material in its zone.

Let us set

$$\frac{m}{Q}=z^3, \quad z^2\tau_e^*=\left(\frac{Q\delta^2}{36\pi}\right)^{1/3}\frac{P}{(1+2\theta)\,\sigma_0}=\tau_0. \tag{16}$$

Then (15) becomes

$$3\tau_0\frac{dz}{dt}=1-z^3, \tag{17}$$

where τ_0 is the limiting value of τ_e^* when $m\to Q$.

If the mean embryo density δ and parameter θ are taken to be constant, then $\tau_0=$ const and the above expression is easy to integrate (Shmidt, 1945):

$$\frac{t}{\tau_0}=\sqrt{3}\,\mathrm{arc\,tg}\frac{1+2z}{\sqrt{3}}+\frac{1}{2}\ln\frac{1-z^3}{(1-z)^3}-\frac{\pi\sqrt{3}}{6}=f(z). \tag{18}$$

The planet mass m tends asymptotically to Q, while the amount of material not used up within the zone $Q-m$ decreases exponentially with time. In the concluding stage of growth where $m\approx Q$ and $\tau_e^*\approx\tau_0$, we obtain, from (15),

$$Q-m=(Q-m_0)\,e^{-(t-t_0)/\tau_0}, \tag{19}$$

where $Q-m_0$ is the amount of unused material at the instant t_0. For the Earth zone for $\theta=3$ and $\delta=5.52$, $\tau_0=17$ million years, while according to Shmidt's formula it should have amounted to about one quarter of a billion years. Within the first billion years of its existence, the Earth had exhausted all the material in its zone. Thus it is entirely out of the question to estimate the Earth's age from the residue of unused matter in its zone with formula (18). The interplanetary material that falls on the Earth today is not a residue of the primary substance of the Earth zone; it is the product of the disintegration of comets and asteroids continuously entering the Earth's zone from parts of the solar system more distant from the Sun. The Earth's age must be evaluated by more direct methods. Thus according to measurements for rocks, meteorites and chemical elements, it is now estimated that the planets are approximately 5 billion years old, i. e., nearly as old as the Sun. A recent estimate, for example, is that of Tilton and Steiger (1965), who obtain the figure 4750 ± 50 million years for the age of the Earth from lead isotope ratios in ancient rocks of the Canadian shield.

In the derivation of formula (18) for growth it was assumed that the planetary zone was closed, or more precisely that the total amount of solid

material in the zone was conserved at all times and that its initial mass was equal to the present mass of the planet (relation (13)). This is an important assumption, and for the giant planets it does not hold, since in the final stage of growth their source zones fused into one open zone (bodies were ejected beyond the solar system). But for planets of the terrestrial group (with the exception of Mars) it is wholly applicable. Bodies with relative velocities corresponding to $\theta = 3$ could not have traveled beyond Mars' orbit, for example, and most of the bodies never reached it.

Formula (18) makes it possible to calculate the duration of the process of planet formation. Owing to the asymptotic character of its vanishing, the choice of concluding instant for this process is arbitrary. In 1954 and 1957, we estimated the growth span up to the instant when a planet reaches 97% of its present mass, i.e., when $z = 0.99$. For $z = 0.99$ the right hand side of (18) is given by $f(0.99) = 6.0$ and thus the planet formation time τ_p as determined in this manner is given by

$$\tau_p = 6\tau_0. \tag{20}$$

For $\theta = 3$ and $\delta = 5.52$ the Earth's growth time is 10^8 years. The variation in the mean density of the planet during the growth process can be allowed for fairly accurately (within 1–2%) by taking the average of the initial and final mean planet density for δ when calculating τ_0 in (16). Then for the Earth one can take $\delta \approx 4.5$ and $\tau_p = 0.88 \cdot 10^8$ years. Within 100 million years the Earth's mass must have grown to 98% of its present mass. A graph showing the rate of growth of the Earth with allowance for the variable δ is given in Chapter 14.

Table 11 gives the characteristic exhaustion times τ_0 of planets of the Earth group (in million of years), as obtained for present values of planet mass and density and for $\theta = 3$ and $\theta = 5$, in accordance with (16). The surface density σ_0 was determined from present planet masses; boundaries between planet zones were taken to be the average of figures obtained by Shmidt and by Gurevich and Lebedinskii from laws governing planetary distances.

TABLE 11

	Mercury	Venus	Earth	Mars
σ_0	1.5	16	10	0.3
τ_0 ($\theta = 3$)	10	5	17	400
τ_0 ($\theta = 5$)	6	3	11	250

The value of σ_0 obtained for Mars is unusually low (0.3 g/cm^2). Assuming that the width of the zone was $2\Delta R$, in accordance with (8), then for $\theta = 3$, $\sigma_0 \approx 1$ g/cm^2. Since within the Jupiter zone a surface density of solid matter of 20–30 g/cm^2 amounts to 3–5 g/cm^2 when translated to silicates, while in the Earth zone $\sigma_0 \approx 10$ g/cm^2, one would expect that within the Mars zone it would be ~ 7–8 g/cm^2. This means that only about 10% of the solid substance in its zone was absorbed by Mars, and the remainder left the zone. Consequently, it becomes meaningless, strictly speaking, to evaluate τ_0 for Mars,

as the material in its zone was, for the most part, not used up by Mars. What happened to Mars is nearly the same as what happened to the asteroids (see Chapter 12). Owing to external perturbations (influx of bodies from the Jupiter zone; see Section 34), velocities in the Mars zone rose far more rapidly than could be expected if the perturbations had been due to Mars alone, and this slowed down its growth very markedly (in expression (2) for the collision cross section, $(1+2\theta)\to 1$).

Since $\tau_p \propto \tau_0 \propto P$, planet growth was on the average more rapid in the inner portion of the cluster than in the outer cluster.

For $z \ll 1$ the growth formula (18) is much simpler:

$$t = 3z\tau_0 = 3\left(\frac{m}{Q}\right)^{1/3}\tau_0. \tag{21}$$

At the early stage the density drop due to exhaustion is insignificant and the radius r increases in proportion with time. Introducing $m_0 = \sigma^3/\rho^{*2}$ in (21) in accordance with (6.6), we obtain the time $t(m_0)$ necessary for the mass of a small body to increase to that of condensations formed by gravitational instability. It is independent of σ and is proportional to $R^{1/2}$. Even in the Mars zone it amounts to only $3 \cdot 10^4$ years (see below):

Zone	Mercury	Venus	Earth	Mars
$t(m_0)$, years	$3 \cdot 10^2$	$2 \cdot 10^3$	$8 \cdot 10^3$	$3 \cdot 10^4$

Consequently, even if any factor prevented the appearance of gravitational instability in the region of the Earth group planets (see Chapter 3), within a fairly short time direct particle growth will have led to the formation inside this region of bodies with dimensions of the order of kilometers. Low relative velocities practically ruled out the possibility of particle fragmentation in collisions. An extremely low gas density (10^{-9} g/cm^3 and less) made possible efficient adhesion, as in cold welding (Levin, 1966).

Chapter 10

ROTATION OF THE PLANETS

28. Critical analysis of earlier research

The problem of the planets' rotation is one of the most difficult in planetary cosmogony. Not unexpectedly, different cosmogonic theories envisaged different solutions. An extensive review of work on planetary rotation was published in 1963 by Artem'ev. Below we consider only the more important theories.

Authors developing the Laplacian hypothesis usually related the planets' direct rotation to the action of the Sun's tidal forces. Poincaré gives the following schematic description of this process (1911). A gaseous ring separates from the central condensation and begins to revolve around the Sun as a rigid body owing to frictional forces. Eventually it becomes unstable and disintegrates. Separate sections of the ring begin to move along circular orbits. When two clusters situated at somewhat different distances from the Sun combine, retrograde rotation sets in, since the inner cluster will travel more rapidly along the orbit than the outer one. However, the tidal forces of the Sun attract the cluster and impart to it a direct rotation of period equal to the period of revolution. Contraction of the cluster due to cooling reduces the tidal forces and increases the rate of rotation.

None of these conjectures is admissible today. Friction inside a ring revolving around the Sun would not cause it to revolve as a rigid body. It would merely move the inner part of the ring closer to the Sun and cause the outer part to move away, rotation remaining Keplerian at all times. The idea of the disintegration of the protoplanetary cloud into small gaseous clusters and their subsequent fusion is equally untenable. Such clusters would be unstable, tending to disintegrate rather than combine. The only way around this difficulty is by having dust — not gaseous — condensations. But dust condensations are small and contract so rapidly that tidal forces could not markedly affect their rotation.

In the planetesimal hypothesis of Chamberlin (1904) and Moulton (1905) it was assumed that the planet acquired direct rotation in the process of growth as a result of the influx of planetesimals. The authors supposed that for the main the planet acquired positive rotational angular momentum only from bodies with the perihelial distance $R_0 < R_p < R_0+r$ and bodies with the distance $R_0-r < R_a < R_0$ at aphelion, where r is the radius of the growing planet and R_0 its distance from the Sun. It was found that, for an orbital eccentricity $e = 0.2$ for the bodies and impact parameter equal to $6.5 \cdot 10^8$ cm, bodies of this class could have imparted the required rotation to the Earth provided their mass was equal to 5.7% of the Earth mass. However, it was

not made clear whether such a high percentage of bodies within such a narrow range of perihelial and aphelial distances is possible.

Hoyle (1946) suggested an explanation for planetary rotation based on the concept of planet growth by accretion (aggregation; see Section 1) of diffuse matter envisioned as a solid medium. The probable capture radius for a planet of mass m and radius r was taken to be half the distance at which the tidal force of the Sun equals the planet's gravitational attraction: $r_a = \frac{1}{2}(m/2M_\odot)^{1/3} R$. The rotational momentum ΔK imparted upon accretion by substance Δm was taken to be $2/5\ \bar{\omega} r_a^2 \Delta m$ (spherically symmetric accretion), where $\bar{\omega} = \omega_e/4$ (see Chapter 6). The period of rotation was found to be 3–4 hours, and would be even less if allowance is made for the concentration of matter toward the center of the planet. Hoyle accepted this very high speed of rotation under the influence of Lyttleton's view of the rotational instability of primary planets and the separating away of satellites. Lyttleton (1960) still maintains this view, but it lacks a firm foundation. On the other hand accretion could have played an important role only in the growth of Jupiter and Saturn, when these had become massive enough to absorb gaseous hydrogen. Recently Hoyle (1960) renounced the application of the accretion mechanism to Earth group planets. If the capture radius in accretion is revised down to 2–3 times less than the value adopted by Hoyle, the interpretation of Jupiter's and Saturn's rotation in terms of accretion theory will become satisfactory.

The theory of planet formation from massive protoplanets (Kuiper, 1951; Fesenkov, 1951) postulated the formation of massive clusters as a result of the onset of gravitational instability in the gaseous component of the cloud. The condensation of the protoplanet was envisioned as a process of collection of material along the orbit of the primordial cluster. To determine the planet's rotation Fesenkov computed the angular momentum of a section of a torus of diameter $2l$ with reference to the axis passing through the center of the section. It was found that the present planetary rotation is obtained only for values of l far smaller than the width of the planetary zone. (The torus diameter should be 70 times smaller than the zone width in the Jupiter zone and 300 times smaller in the Earth zone.) The excessively high planetary rotation yielded by the theory of massive gaseous protoplanets further compounds the serious difficulties which it presents (Ruskol, 1960) and which have defied solution.

Gurevich and Lebedinskii (1950), in their approach to the problem of planetary rotation, considered the rotational angular momentum of dust condensation whose fusion led to the formation of the planets. They obtained an expression for the angular momentum of the condensations in the form $k_0 M_n (M_n/M_\odot)^4$, where k_0 is the specific orbital angular momentum and M_n the planet mass. From this it was concluded that the rotational angular momentum of a planet should be equal to the orbital angular momentum multiplied by a certain function of the planet mass. This result was illustrated by an empirical dependence which fits all planets other than Saturn and Neptune. This conclusion essentially rests on the implicit assumption that condensations combining with each other suffer central collisions. In noncentral collisions, in addition to the proper rotational angular momenta of the combining condensations, one must also take into account the considerably greater angular momenta associated with their relative orbital motion.

Looking at the problem of planetary rotation, Shmidt (1957) started with an analysis of the general laws governing the process of fusion of material into a planet. He wrote down conditions of energy and angular momentum conservation for transition from a particle cloud to a planet. The angular momentum of the particles situated in the planetary zone changes into the orbital and rotational angular momenta of the planet. The smaller the orbital angular momentum, i. e., the smaller the radius of the planet's orbit, the greater its rotational angular momentum should be. But as the orbital radius of the planet decreases, so does its orbital energy while as a result, according to Shmidt, the thermal loss of energy in the process of planet formation increases. Hence Shmidt's major result: since the energy losses in this process are large, the planetary rotation should be direct. The mathematical formulation of this result reduces to the following.

Consider a cloud of particles traveling around the Sun along circular orbits lying in the same plane. From these particles is formed a planet of mass m and orbital radius R_0. Let R_e and R_m be the mean distances between cloud particles averaged over the energy and the angular momentum respectively:

$$\frac{1}{R_e}=\frac{\int\limits_{R_1}^{R_2}\frac{\varphi(R)}{R}\,dR}{\int\limits_{R_1}^{R_2}\varphi(R)\,dR}=\frac{1}{m}\int\limits_{R_1}^{R_2}\frac{\varphi(R)\,dR}{R},$$

$$\sqrt{R_m}=\frac{\int\limits_{R_1}^{R_2}\sqrt{R}\varphi(R)\,dR}{\int\limits_{R_1}^{R_2}\varphi(R)\,dR}=\frac{1}{m}\int\limits_{R_1}^{R_2}\sqrt{R}\varphi(R)\,dR, \tag{1}$$

where $\varphi(R)$ is the mass distribution function of the particles over the distance from the Sun, and R_1 and R_2 are the boundaries of the zone of the planet under consideration.

One can prove that $R_m > R_e$ at all times (just as the mean square is always greater than the simple mean). Therefore if the condition

$$R_0 < R_e \tag{2}$$

holds, the following inequality should be satisfied:

$$R_0 < R_m, \tag{3}$$

i. e., the rotation should be direct, since the planetary orbital momentum, proportional to $\sqrt{R_0}$, is smaller than the angular momentum of the cloud particles, which is proportional to $\sqrt{R_m}$.

The conditions under which relation (2) is fulfilled were not analyzed. Shmidt believed that "we cannot determine the sum of these losses quantitatively, but there is no doubt that the losses are large." He proposed further that these same factors were responsible for the direct revolution of most of the planetary satellites and that the retrograde revolution of distant satellites was due to nonfulfillment of condition (2).

Let us analyze more carefully Shmidt's equations for the energy and angular momentum balance and his condition for direct planetary rotation (Safronov, 1962c). We will adopt the following notation: U_0 — potential energy of planet with reference to the Sun; U_p — potential energy of planet as a sphere; U_e — potential energy of particle interaction; E_0 — orbital energy of planet (potential plus kinetic); E_r — kinetic energy of planetary rotation; E_t — loss in energy of mechanical motion resulting from its transformation into other forms of energy, such as warming, radiation, phase transitions, etc.; M — Sun's mass; m — mass of growing planet; R_0 — radius of planetary orbit with reference to Sun; K_0 — orbital angular momentum of planet; K_r — rotational angular momentum of planet. The balance equations derived by Shmidt then become

$$-\frac{GM}{2}\int_{R_1}^{R_2}\frac{\varphi(R)}{R}\,dR+U_e=E_0+U_p+E_r+E_t, \tag{4}$$

$$\sqrt{GM}\int_{R_1}^{R_2}\sqrt{R}\,\varphi(R)\,dR=K_0+K_r, \tag{5}$$

where

$$E_0=-\frac{GMm}{2R_0},\qquad K_0=m\sqrt{GMR_0},\qquad m=\int_{R_1}^{R_2}\varphi(R)\,dR. \tag{6}$$

According to Shmidt, the first term in the left-hand side of (4) represents the sum of the kinetic and potential energy of particles moving round the Sun along circular orbits, while the term in the left-hand side of (5) represents their total angular momentum with reference to the Sun. These terms were written down for the Sun's gravitational field alone and do not allow for gravitational fields induced by particles.

Artem'ev (1963) has pointed out that these balance equations fail to take account of the energy and angular momentum of the Sun, which after the planets were formed ceased to lie at the center of gravity of the system. Yet the orbital angular momentum of the Sun is several times greater than the rotational momentum of a planet. In (4) – (6) E_0 and K_0 were taken for the planet's motion with reference to the Sun. In reality one should take the sum of E and K for planets and Sun with reference to the center of gravity of the system. In the presence of numerous planets the Sun's motion becomes very complicated. Rigorous balance equations would have to be written down simultaneously for all bodies in the solar system, and they could not yield concrete results for isolated individual planets. But analysis is possible if one confines oneself to a single planet traveling around the Sun. Denoting by E_c and K_c, respectively, the sum of the orbital energy and the sum of the orbital momenta of the planet and Sun, with reference to their center of gravity, it is easy to find that

$$E_c=-\frac{GMm}{2R_0},\qquad K_c=m\sqrt{\frac{GmR_0}{1+m/M}}, \tag{7}$$

where R_0 is the distance of the planets from the Sun (not from the center of gravity). Quantities E_c and E_0 are identical while K_c differs from K_0, given in (6), by an amount $\approx -\frac{1}{2}\frac{m}{M}K_0$.

Corrections to the left-hand side of equations (4)–(5) due to allowance for the gravitational field of particles scattered throughout the planetary zone and having total mass m will be of the same order. In the first equation the correction to the kinetic energy will be $\approx -U_e/2$. Combining it with U_e, we denote the total correction by $\lambda_1 \frac{m}{M} E_0$. The correction to the left-hand side of the second equation will be denoted by $\frac{\lambda_2}{2} \frac{m}{M} K_0$. Obviously, $\lambda_1 \sim \lambda_2 \sim 1$. Then, replacing the integrals in accordance with (1) by R_e and R_m respectively, we obtain

$$\frac{1}{R_e} = \left(1 - \varepsilon - \lambda_1 \frac{m}{M}\right) \frac{1}{R_0}, \tag{8}$$

$$\sqrt{R_m} = \left[1 + k - \frac{1}{2}(1 + \lambda_2) \frac{m}{M}\right] \sqrt{R_0}, \tag{9}$$

where

$$\varepsilon = \varepsilon_p + \varepsilon_t + \varepsilon_r = \frac{2R_0}{GMm}(U_p + E_t + E_r), \tag{10}$$

$$k = \frac{K_r}{m\sqrt{GMR_0}}.$$

Eliminating R_0, we obtain

$$2k \approx \frac{R_m - R_e}{R_e} + \varepsilon + \lambda_3 \frac{m}{M}, \tag{11}$$

where $\lambda_3 = 1 + \lambda_1 + \lambda_2 \sim 1$.

The ratio $(R_m - R_e)/R_e$ depends chiefly on the width of the source zone and only weakly on the form of $\varphi(R)$. For the Earth zone it is of the order of 10^{-2}. There is no basis for assuming that $\varepsilon < 0$ (see below). In precisely the same way, $\lambda_3 > 0$. The right-hand side is found to be positive and thus the rotation must be direct. Formally the problem would appear to be solved. In reality this is not so, as expression (11) yields an inadmissibly high value of k. For the Earth it gives a value which is 10^4 times faster than the actual rotation (the ratio of rotational to orbital angular momentum for the Earth is given by $k \approx 3 \cdot 10^{-7}$).

The reasons for this result are to be found in a defect of the scheme itself. It seems natural to suppose that the particles traveled originally along circular orbits. For small body masses gravitational perturbations must have been weak and the particles moved along orbits that were nearly circular. As the planet grew the deviation of body orbits from circular orbits increased, and all bodies inside the zone were able to combine into a single planet. The balance described above would be valid if the planetary zone remained closed at all times. That this supposition is inadmissible is shown by the exceedingly fast rotation one then obtains. As eccentricities increase owing to encounters between bodies and planet and among the bodies themselves, some bodies travel out beyond the outer boundary of the zone and remain there (becoming "stuck"), removing excess momentum from the zone; others cross the inner boundary, removing momentum which is less than the mean momentum. A simultaneous influx of bodies takes place into

the planet zone from the outside. These exchange processes are not compensatory and as a result the total angular momentum of matter inside the zone, as well as the total energy, including thermal losses, does not remain constant.

The statement of the problem becomes more valid if we allow for the eccentricites and inclinations of the orbits of bodies and particles and include in the function $\varphi(a)$ only those that actually fall on the planet. The difficulties stemming from the fact that the zone is open can then be circumvented to a large extent by considering not the total balance between initial and final states but rather a "differential" balance, i. e., the balance for specified values of m, e and i and for growth of the planet mass by a small quantity Δm. The exchange of angular momentum and energy between bodies landing on the planet and bodies that do not land on it may especially be disregarded, since it is this exchange that causes the bodies to acquire the orbital eccentricities e and inclinations i, which we are taking as initial data. The influence on the planet's motion of bodies not landing on it is apparently negligible, although it is not inconceivable that while traveling along its circular orbit the planet experienced very limited retardation, and therefore reduction in R, due to encounters with bodies whose centrode velocity, as is known, is slightly smaller than the circular velocity.

Let us assume for simplicity that the orbital eccentricity and inclination to the plane of the planet's orbit are the same for all bodies and particles. Quantities a, e and i will be understood to be mean unperturbed elements describing the motion of a body that is not in the process of encountering other bodies. Let the mass imparted by bodies and particles with orbital semimajor axes lying between a and $a+da$ to the planet be $\Delta m\varphi(a)\,da$. Then $\int_{a_1}^{a_2}\varphi(a)\,da=1$. Further, let the planet move along a circular orbit ($a_0=R_0$). Instead of equations (4) and (5) we obtain

$$\Delta m\int_{a_1}^{a_2}\frac{\varphi(a)\,da}{a}-\Delta\left(\frac{m}{a_0}\right)\int_{a_1}^{a_2}\varphi(a)\,da\left(1-\lambda_1\frac{m}{M}\right)=-\frac{2}{GM}\Delta(U_p+E_i+E_r),$$

$$\Delta m\int_{a_1}^{a_2}\sqrt{a(1-e^2)}\cos i\quad\varphi(a)\,da-\Delta(m\sqrt{a_0})\int_{a_1}^{a_2}\varphi(a)\,da\times$$
$$\times\left[1-\frac{1}{2}(1+\lambda_2)\frac{m}{M}\right]=\frac{\Delta K_r}{\sqrt{GM}}. \tag{12}$$

We further have

$$\Delta\left(\frac{m}{a_0}\right)=(1-\gamma)\frac{\Delta m}{a_0},\quad \Delta(m\sqrt{a_0})=\left(1+\frac{\gamma}{2}\right)\sqrt{a_0}\Delta m,$$

where

$$\gamma=\frac{d\ln a_0}{d\ln m}.$$

Next, applying notation (1) to the orbital semimajor axes instead of the orbital radii, we obtain the following expression from (12), by analogy with (9):

$$\frac{1}{a_e}=\left[(1-\gamma)\left(1-\lambda_1\frac{m}{M}\right)-\varepsilon'\right]\frac{1}{a_0},$$

$$\sqrt{a_m(1-e^2)}\cos i=\sqrt{a_0}\left\{\left(1+\frac{\gamma}{2}\right)\left[1-\frac{1}{2}(1+\lambda_2)\frac{m}{M}\right]-k'\right\}, \tag{13}$$

where

$$\varepsilon'=\frac{2a_0}{GM}\frac{d}{dm}(U_p+E_t+E_r),\quad k'=\frac{1}{\sqrt{GMa_0}}\frac{dK_r}{dm}. \tag{14}$$

Eliminating a_0, we find that

$$a_m(1-e^2)\cos^2 i=\left[(1-\gamma)\left(1-\lambda_1\frac{m}{M}\right)-\varepsilon'\right]\times$$
$$\times\left[\left(1+\frac{\gamma}{2}\right)\left\{1-\frac{1}{2}(1+\lambda_2)\frac{m}{M}\right\}+k'\right]^2 a_e. \tag{15}$$

The quantity γ characterizing the variation in the orbital radius of the planet embryo can be evaluated from the second equation in (12), setting $\varphi(a)=ca^{-n}$ and $\Delta K_r=0$ owing to its smallness. Calculations indicate that for admissible values of n the value of γ is of the order of e^2. In (15) γ yields only terms of the order of e^4 and higher powers. Retaining only infinitesimals of the second order in e and i and first order in k' and m/M, we obtain, by analogy with (11),

$$\frac{a_m-a_e}{a_e}\approx e^2+i^2+2k'-\varepsilon'-\lambda_3\frac{m}{M}. \tag{16}$$

Comparing (16) and (11) we see that when the eccentricites and inclinations of the orbits are taken into account, there appear additional terms e^2 and i^2 of the same order as the rest. Hence they make a substantial difference to the result. Consequently a scheme based on circular particle orbits is clearly inadmissible.

To determine the left-hand side of (16) we introduce the dimensionless distance

$$x=\frac{a-a_1}{a_1}$$

and express the distribution function $\varphi(a)$ in terms of x, retaining only second order terms:

$$\varphi=\varphi_1(1+c_1x+c_2x^2). \tag{17}$$

Introducing this expression for φ into (1), where R should be replaced by a throughout, we can find a_m and a_e. Calculations show that up to the third order in x expression (16) will be independent of c_1 and c_2:

$$\frac{a_m-a_e}{a_e}=\frac{x_2^2}{16}(1-x_2+\ldots). \tag{18}$$

The value x_2 corresponds to the outer boundary of the zone (a_2). For the present-day Earth one can take $x_2\approx 0.6$ and $e\approx 0.2$. Since

$$a_1=\frac{R_0}{1+e},\qquad a_2=\frac{R_0}{1-e}, \tag{19}$$

for small e

$$x_2 = \frac{a_2 - a_1}{a_1} = \frac{2e}{1-e} \approx 2e, \qquad \frac{a_m - a_e}{a_e} \approx \frac{e^2}{4}.$$

Introducing this value into (16), we obtain

$$\varepsilon' \approx \frac{3}{4} e^2 + i^2 + 2k' - \lambda_3 \frac{m}{M} \tag{20}$$

or

$$E_t' + E_r' \approx -U_p' + \frac{1}{2} V_c^2 \left[\frac{3}{4} e^2 + i^2\right] + k' V_c^2 - \frac{\lambda_3}{2} \frac{m}{M} V_c^2. \tag{21}$$

This expression can be written in more explicit form by introducing the body velocity V (with reference to the planet) unaffected by the latter's gravitation (velocity before encounter). The velocity component perpendicular to the plane of the planet orbit $v_z \approx i V_e$, where V_e is the circular velocity and the component in the orbital plane $v_{R\varphi} \approx e V_e \cdot \sqrt{1 - \frac{3}{4} \cos^2 \varphi}$, where φ is the angular distance of the body from perihelion at the instant of encounter. Let $v_{R\varphi}^2 = \lambda_4 e^2 V_e^2$. Then

$$v^2 \approx (\lambda_4 e^2 + i^2) V_e^2 \tag{22}$$

and from (21) we find that

$$E_t' + E_r' \approx -U_p' + \frac{v^2}{2} + k' V_e^2 + \frac{1}{2}\left(\frac{3}{4} - \lambda_4\right) e^2 V_e^2 - \frac{\lambda_3}{2} \frac{m}{M} V_e^2. \tag{23}$$

The exact value of λ_4 is difficult to compute. It requires knowledge of the density distribution $\varphi(a)$ inside the cluster. In any event λ_4 is close to 3/4 and the next to last term in (23) is at least one order smaller than v^2. If one takes the simple mean of $\left(1 - \frac{3}{4} \cos^2 \varphi\right)$, then $\lambda_4 = 5/8$ and the next to last term is equal to $v^2/16$.

Analysis of this equation leads to the following conclusions.

1. The fusion into a planet of bodies traveling along elliptical orbits does not lead to the inadmissibly rapid rotation which is obtained for the fusion of bodies moving along circular orbits.

2. To order of e^2 (the accuracy with which expression (23) was obtained) the thermal losses incurred when particles strike the planet surface are equal to the sum of the potential energy released in the fall and the kinetic energy of the particle before encounter with the planet (E_r' is three orders smaller than U_p'). This is the value we took for the losses when evaluating the Earth's primordial temperature (1959).

3. Expression (23), which was derived from the balance equations, contains two unknown quantities: the velocity of rotation of the planet and the thermal losses in the accumulation process. The balance equations are not sufficient for solving the problem of the planets' rotation. Only by analyzing a definite collision mechanism will it be possible to determine both rotation and losses with the aid of these equations.

4. If the term E'_r in (23) characterizing the energy of rotation had been significantly smaller than the term $k'V_e^2$, then k' would have increased with increasing E'_t and one could have expected direct rotation with high thermal losses. The term with k' is the smallest in (23), and it is practically impossible to evaluate it, as it appears in (23) among quantities which are four orders larger. Consequently, no conclusion regarding the direction of planetary rotation can be drawn from (23). For the present-day rotational velocity of the Earth the term $k'V_e^2$ is 2.5 orders smaller than E'_r. The relationship between rotation and thermal losses should therefore be inferred from E'_r and E'_t rather than from k' and E'_t. These quantities appear throughout as a sum. For specific eccentricities of body orbits, the faster the planet's rotation, the smaller should be the thermal losses in the accumulation process. This result is qualitatively understandable: acceleration of the rotation intensifies as the number of particles striking in the direction of rotation increases, and as the number striking counter to the direction of rotation decreases (i. e., as the mean velocity of particle impacts and hence the thermal losses decrease). Incidentally, this result also holds in the case considered by Shmidt of body motion along circular orbits. In his balance equations the energy loss also appears as a sum together with the rotational energy.

Thus when one allows for the planet's rotational energy, which in the balance equations plays a considerably larger role than the planet's rotational momentum, one is led to conclude that thermal losses decline as the velocity of rotation increases. The rotational energy seems to arise at the expense of thermal losses.

It is possible to determine what planetary rotation corresponds to maximum thermal losses. Let us assume that a certain set of particles and bodies combines to form a planet in two ways characterized by different thermal losses. In the first case the planet is formed on a circular orbit of radius R and in the second, on one of radius $R+\delta R$. The total angular momentum should be the same in both cases, and therefore changes in the planet's orbital momentum are compensated by changes in its rotational momentum

$$\delta K_r = -\delta K_0 = -\frac{m}{2}\sqrt{\frac{GM}{R}}\,\delta R,$$

which leads to a change in the rotational energy

$$E_r = \frac{1}{2} I_r \omega_r^2; \qquad K_r = I_r \omega_r; \qquad \delta E_r = I_r \omega_r \delta\omega_r = \omega_r \delta K_r;$$

$$\delta E_r = -\frac{m}{2}\omega_r V_e \delta R.$$

The change in orbital energy

$$\delta E_0 = \frac{GMm}{2R^2}\delta R = -\frac{\omega_e}{\omega_r}\delta E_r$$

will be much smaller than δE_r in view of the smallness of the orbital angular velocity ω_e compared with the rotational velocity ω_r. Therefore the change in overall mechanical energy

$$\delta E_0 + \delta E_r = -\frac{m}{2} V_e (\omega_r - \omega_e) \delta R \tag{24}$$

is practically determined by the change δE_r. Next,

$$E_0 + E_r + E_t = \text{const}; \qquad \delta E_t = -\delta E_0 - \delta E_r.$$

Therefore if the thermal losses increase (i. e., if the mechanical energy decreases), for $\omega_r > \omega_e$ one should have $\delta R > 0$ and the velocity of rotation will decrease.

Energy losses are maximum when the sum of the orbital and rotational energies is minimum. For this to happen it is necessary that $\delta E_0 + \delta E_r = 0$, or, according to (24), $\omega_r = \omega_e$. Consequently, thermal losses are maximum when the planet rotates about its axis with the velocity of rotation of the cluster itself, i. e., when it does not rotate relative to the cluster. For maximum thermal losses the rotation is found to be direct but excessively slow compared with actual rotation of the planets. The rotation relative to the cluster increases with decreasing thermal losses. From the standpoint of the loss it is unimportant in which sense the rotation proceeds if the latter is reckoned from ω_e, since it can be shown that the losses are the same for the rotational velocities $\omega_e + \Delta\omega$ and $\omega_e - \Delta\omega$.

An attempt to attribute the planets' direct rotation to large thermal losses during their formation was also made by Khil'mi (1951) based on a few general relations of mechanics. However, a closer analysis of these relations by the same author (1958) indicated that it does not inevitably follow from them that the planets' rotation was direct.

It should be stressed that the objections mentioned above do not affect the foundation of Shmidt's theory regarding the formation of the planets by the accumulation of solid bodies and the considerable role of thermal losses in this process. If there had been no thermal losses collisions between bodies would have been absolutely elastic and their accumulation into planets would have been impossible. Our objections are directed only at the idea that the planets' direct rotation is also due to large thermal losses. In reality both the planets' direct rotation and a certain loss of energy in the process of accumulation were due to concrete conditions of collision between combining bodies, i. e., to the basic laws governing their motion. If the planets had formed in a nonrotating cluster, for example, they would not have acquired regular rotation although losses would have reached values of the same order.

A substantially different explanation of the planets' rotation was suggested recently by Artem'ev and Radzievskii (1963, 1965). The authors conjecture that the planet acquired nearly all of its rotational momentum not from bodies falling on it directly, but from bodies originally captured by the planet as a result of inelastic collisions in its gravity field, and subsequently falling on it. In one variant the velocities of the bodies after collision were assumed to be equal to the circular Kepler velocity; in another variant they were taken to be equal to the corresponding rigid rotation of a Hill region around the Sun. This idea of a two-stage process of fall on the Earth enabled them to interpret r in the expression for the angular momentum acquired by the planet ($\frac{1}{5}\omega_e r^2 dm$ in the first variant; see Chapter 6) not as the planet radius, but nearer the radius of the largest closed Hill surface (i. e., at least two

orders greater). It was found that for the planets to acquire their present angular momentum it is necessary that such capture be experienced, depending on the scheme, by a few percent of the mass to nearly the entire mass of the planet (the latter would seem to be more probable). This is much more than the mass of all the satellites, which according to modern views were formed from material in the same planetary cluster captured by the gravitational field of the planet as a result of inelastic collisions among bodies and particles in its vicinity (Ruskol, 1960). Artem'ev and Radzievskii's hypothesis thus leads to important consequences regarding the character of the formation process and the evolution of satellite clusters around the planets. To test the hypothesis one would have to calculate directly the amount of material captured by the planet and its angular momentum with reference to the planet. This in turn requires further progress in accumulation theory.

29. Methods for solving the problem

Of the two conservation equations (4) and (5) employed by Shmidt, only the equation for angular momentum conservation is directly related to the problem of the planets' rotation. The equation of energy conservation only makes it possible to evaluate thermal losses in the process of accumulation after the planet's rotation has been determined from the angular momentum equation. Since the planet's rotational momentum amounts to only one millionth of its orbital momentum, the form (5) and (12) of the angular momentum conservation equation is extremely inconvenient to use in quantitative estimates. It is far more expedient to write the equation in a coordinate system directly related to the planet.

Consider the collision with a planet of mass m and radius r of a particle Δm having, at the instant of impact, a velocity v' with reference to the planet, directed at an angle θ to the inner normal at the point of incidence. Its total angular momentum at impact is conserved with reference to the center of gravity of the system $(m, \Delta m)$. From this conservation law we obtain the increment in angular momentum of the planet when struck by the body Δm:

$$\Delta K_r = \frac{m\Delta m}{m+\Delta m} v' r_c \sin\theta + \frac{m\Delta m}{m+\Delta m} v' (r - r_c) \sin\theta = \frac{m\Delta m v' r}{m+\Delta m} \sin\theta, \qquad (25)$$

where r_c is the distance of the center of gravity after incidence of Δm from the original center of the planet. It can be shown that this equation is equivalent to the second equation of (12) and that it can be derived from it by simple vector transformations. Since ordinarily $\Delta m \ll m$, it can be written in the simpler form

$$\Delta K_r \approx \Delta m v' r \sin\theta, \qquad (25')$$

i. e., the angular momentum acquired by the planet when the body Δm falls on it is equal to the angular momentum of Δm with reference to the planet at the time of impact. Thus the problem of planetary rotation can be reduced to a statistical discussion of a limited three-body problem (Sun, planet,

particle of small mass). Each particle imparts to the planet an angular momentum directed basically at random. The problem is to obtain the mean of the incidence of many particles. An approximate discussion of the problem with the introduction of a sphere of action and replacement of the three-body problem by two two-body problems has been carried out by Artem'ev.

It seems that any real hope of solving this complex problem lies chiefly with numerical methods. Individual attempts at a numerical solution are already under way. The results obtained are encouraging, though as yet no general conclusions may be drawn. Kiladze (1965) has computed quasi-circular orbits on a computer for a limited circular three-body problem (the orbits start in the vicinity of the libration point L_3, behind the Sun, and end at the planet's surface). A one-parameter family of these orbits was selected with the aid of a condition imposed on the initial coordinates and velocities, enabling the author to simplify his equation to some extent. The rotational momentum imparted to the planet by particles moving along these trajectories was found to be negative for a planet mass m equal to the mass of Jupiter. Kiladze believes that for $m / M_{\odot} < 1/1500$ the angular momentum ought to be positive. He has initiated calculations for other classes of initial particle orbits with a view to estimating the mean rotational momentum imparted to the planet by the entire set of particles falling on it.

Giuli (1968) has computed several tens of families of trajectories on a computer, beginning at great distances from the planet in the form of ellipses with definite values of a and e in each family and ending at the planet's surface. For most families the angular momentum imparted to the planet was found to be negative, but the overall angular momentum was positive thanks to the particularly effective contribution of certain families. According to Giuli, the period of rotation of the Earth calculated in this manner is 7.3 hours. This work is very interesting, but for 15% of the trajectories, which were more complex, calculations were not carried through and thus it is not clear how accurate these results are.

For a complete solution of the problem of the origin of the planets' rotation it is also necessary to evaluate the amount of material reaching the planet from the satellite cluster and the rotational momentum it imparts to the planet (see Section 28). One factor leading to settling of this material is the growth of the planet's radius and mass and the corresponding shrinking of the orbits of all particles in the cluster; another is the small mean angular momentum of the captured material. In order for the material settling down from the cluster to impart the entire necessary angular momentum to the planet, its mass must amount to a significant fraction of the planet mass. It seems that the probability for this is low.* According to Ruskol, the

* The fact that the orbits of close regular satellites coincide with the equatorial plane of the planet is not decisive proof in favor of this particular method of acquisition of angular momentum by the planet. The planet's equatorial contraction leads to precession of the orbits of the particles and bodies in the satellite cluster with reference to its equatorial plane (Goldreich, 1965a). Owing to differences in the periods of precession of individual bodies, the cluster, originally characterized on the average by a certain inclination ϵ to the planet's equatorial plane, expands so that fairly soon the body orbits lie on both sides of the equatorial plane (within angle $\pm\epsilon$) and the latter becomes the mean plane of the cluster. As a result of inelastic collisions among the bodies the cluster flattens out, but it lies within the equatorial plane. Any deviation from this plane, such as the prevailing incidence of material in the plane of the ecliptic, will again lead to thickening of the cluster (with the aid of precession) and its mean plane will again tend to the equatorial plane of the planet. With increasing distance from the planet, the plane with reference to which the orbital precession takes place shifts away from the equatorial plane and approaches the plane of the planet's orbit.

density of matter was highest in the inner part of the cluster. It is therefore not excluded that a considerable fraction of the rotational momentum of the planet was imparted to it by the material of the satellite cluster. It may be that the problem of the planets' rotation will prove to be closely related to that of the formation and evolution of satellite clusters.

In the absence of a rigorous solution to the problem of planetary rotation, interest attaches to qualitative considerations regarding possible laws governing rotation. Certain results can be obtained by exploiting the concept of asymmetry of the impacts of falling bodies and particles (Safronov, 1960a). Within the limits of the two-body problem the rotational momentum imparted to a planet of mass m and radius r by an individual particle m_i can be written as

$$K_i = \beta_i m_i v r,$$

where v is the relative particle velocity before encounter with the planet, $\beta_i r = l_i$ is the impact parameter, and β_i lies between 0 and $\sqrt{1+2\theta}$. Despite the fact that the direction of the vector $\mathbf{K}_i$ is basically random, due to the presence of a third body (the Sun) and to the rotation about it of a cluster of particles the mean value of $\mathbf{K}_i$ is nonzero, i. e., there exists a systematic angular momentum component. Let us denote it by K_1. Then

$$dK_1 = \beta v r dm,$$

where the coefficient β characterizes the asymmetry of the impacts. If we assume that β remains constant throughout the process of planet growth, then for $v = \sqrt{\frac{Gm}{\theta r}} \propto r$ we obtain

$$K_1 \propto \int r^2 dm \propto m^{5/3} \propto r^2 m \tag{26}$$

and since $K_1 = \frac{2}{5}\mu\omega r^2 m$, $\omega \approx$ const. Then the angular velocity of rotation of the planet will on the average remain constant during its growth process.

If we assume further that β is independent of the planet's distance from the Sun we find that the angular velocities of rotation should be equal for all planets (no allowance being made for the variation of the planet density δ and the parameter θ). However simplified this assumption may be, the result is not very far from reality. It is well known that with the vast differences in planet masses, the angular velocities of planetary rotation vary comparatively little. If we allow for the parameter θ, the inhomogeneity coefficient of the planet μ and its density δ, which appear in the expression for K, we then obtain (regarding them as constant throughout the process of planet growth)

$$\omega \propto \sqrt{(2+1/\theta)\delta}/\mu. \tag{27}$$

For $\theta \geqslant 3$ the dependence of ω on θ is very weak and can be disregarded. The variations in ω due to variation in δ and μ are small. Quantity ω increases slowly with m owing to the increase in δ and decrease in μ.

The fact that the values of δ and μ averaged over the entire period of planetary growth (and not the present values) must be taken in (27) makes it

somewhat complicated to compare (27) with actual data for the planets. Moreover, most of the planets have their own characteristic features which are not taken into account in the above simplified growth scheme. The rotation of Mercury, Venus and the Moon is completely braked by tides. Tides slow the Earth's rotation substantially and partially brake Neptune's rotation. Although Triton is more massive than the Moon and its distance from Neptune is somewhat less than the Moon's distance from the Earth, it, unlike the Moon, is drawing closer to Neptune: earlier it was farther from Neptune than it is today. This is why the retardation of Neptune's rotation could not have been considerable. Jupiter and Saturn contain more gas than solid substance, but the specific angular momentum acquired in the accretion of the gas may have been different from that acquired in the accumulation of the solid substance. Uranus has an obvious anomaly: its axis of rotation is inclined at an angle of 98°, because its random component of rotation was greater than the systematic component (see Chapter 11). That is, the systematic component is simply unknown. Pluto wholly fails to conform to the general pattern. Its mass, moreover, is still not known. This leaves only two planets with which to establish the laws governing rotation — Mars and Neptune. This is clearly insufficient in view of the fact that a planet's rotation may depend not only on its mass but also on its distance from the Sun.

MacDonald (1964) constructed an empirical function for the dependence of specific rotational momentum on the mass of the planets ($k(m)$), approximating it by a power function with exponent close to 0.8. Assuming that the Earth satisfied this dependence primordially, he obtained an initial period of rotation of 13 hrs (this corresponds to an initial distance between Moon and Earth of about 40 Earth radii). Hartmann and Larson (1967), introducing asteroids with known periods of rotation, approximated an inclusive dependence $k(m)$ for all bodies by a power function with exponent 2/3, which corresponds to invariance of the period of rotation (disregarding differences in δ and μ). For the Earth this dependence yields an angular momentum equal to the total angular momentum of the Earth-Moon system. From this the authors infer that the Earth and Moon originally constituted a single body with a period of rotation of 4—5 hrs. A similar function $k(m)$ was obtained by Fish (1967).

The construction of a single function $k(m)$ for planets and asteroids would be meaningless if the asteroids' rotation had altered substantially since they were first formed. According to Hartmann and Larson, collisions between relatively large asteroids (over 10^{17} g) were exceedingly rare and could not have altered their rotation appreciably. They see confirmation of this in Anders' data (1965) on the absolute size distribution of the asteroids, which in Anders' view show that the largest bodies underwent little fragmentation, and in remarks by Alfvén (1964) to the effect that if the number of collisions had been large, the rotational energy of asteroids of different mass would on the average have been the same (equipartition), which is not confirmed by observation. But Hartmann and Larson's reasoning is not convincing enough. Anders assumes that asteroids brighter than the ninth absolute magnitude are "primary;" weaker ones underwent fragmentation. But on the average velocities of rotation were the same in both groups. Consequently, the primordial asteroids somehow managed to acquire large rotational velocities characteristic of colliding asteroids. The tendency to even distribution of the energy of rotation could only have occurred in a system of bodies with

absolutely elastic collisions without fragmentation. In the presence of dissipative processes the energy of small bodies decreases much faster. Moreover, in collisions between asteroids with velocity ~ 5 km/sec, the smaller one should disintegrate and drop out from among those of known rotation.

Lastly, the special conditions that prevented the asteroids from combining into a single planet during the process of growth could not have failed to leave their mark on the asteroids' rotation. In the first place, the large relative velocities of the bodies in the concluding phase, which caused the asteroids to stop growing (see Section 34), must also have led to a higher velocity of regular rotation ($\theta < 1$ in formula (27) for the angular velocity). Secondly, owing to bodies flying into the asteroid zone from the Jupiter zone and to the considerable relative velocities of the bodies inside the asteroid zone, there was no great difference between the mass of the planet embryos m and the masses of other large bodies m_1 (see Section 26). This must have contributed considerably to the asteroids' random component of rotation (Section 30), and the latter may have exceeded the regular component.

Thus there is no physical basis for extending to the asteroids the function $k(m)$ obtained for the planets. A single power function was obtained by Hartmann and Larson at the expense of a considerable reduction in accuracy for the planets. Thus with regard to the initial period of rotation of the Earth, MacDonald's approximation is unquestionably the one to be preferred.

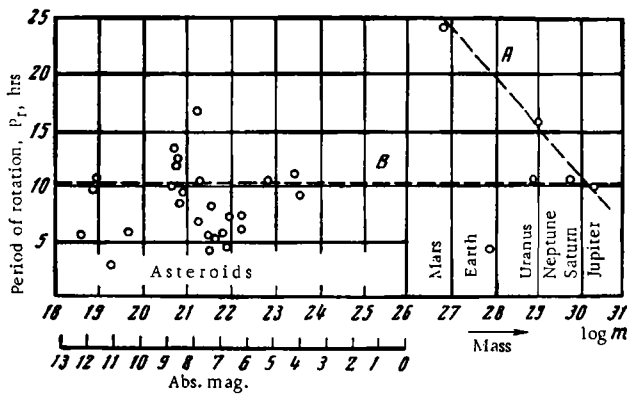

FIGURE 7. Period of rotation of planets and asteroids as a function of their masses. The straight line A passes through all planets besides the Earth and Uranus. The straight line B (same angular velocity) passes through the the three giant planets and asteroids with wide dispersion of points.

But MacDonald's approximation is not the only possible one. In Figure 7 the values of the period of rotation P_r are plotted as a function of $\log m$. The points fit more comfortably on the straight line than they do in MacDonald's graph. For bodies with the mass of the Earth, the dependence yields a period of rotation of about 20 hrs. Linear approximation on a $\log P_r - \log m$ graph gives approximately the same value for the period, and the dispersal of points is 1.5 times less than on the $\log k - \log m$ graph. The divergence in the values of P_r for the Earth on different graphs can be explained with the

aid of the relation $k \propto \mu\omega r^2 \propto \mu\delta^{-5/3}m^{2/3}/P_r$. The planet's coefficient of inhomogeneity μ is approximated by a power of m without substantial distortion. But due to the differences in chemical composition between planets, their densities differ widely independently of their masses. The presence of the factor $\delta^{-5/3}$ leads to various deviations in the actual values of P_r and k from the monotonic (power) function of m. Taking relation (27), which was obtained from considerations relating to the asymmetry of impacts from falling bodies, we obtain

$$P_r \propto \mu\delta^{-1/3}, \quad k \propto \delta^{-1/6}.$$

The Earth is denser than other planets from which the functions $k(m)$ and $P_r(m)$ were determined. Introduction of the appropriate correction increases the value obtained by MacDonald for the initial period of rotation of the Earth from 13.1 to 14.4 hrs and reduces the value obtained from the function $P_r(m)$ from 20 to 15 hrs. These values of P_r correspond to an initial distance between Moon and Earth of about 45 Earth radii.

Chapter 11

THE INCLINATIONS OF THE AXES OF ROTATION OF THE PLANETS

30. Evaluating the masses of the largest bodies falling on the planets from the inclinations of the axes of rotation of the planets

One of the most serious difficulties encountered in the development of a theory of planetary accumulation is the scarcity of observational data capable of serving as checks on different parts of the theory. The data that exist are limited chiefly to the laws of motion and planetary composition. Any opportunity to make use of these data is very important for the theory. It was discovered by the author (Safronov, 1960a) that the inclinations of the planets' axes of rotation are related to the random character of the impacts of individual bodies falling on the planets during the accumulation process, and that the sizes of the largest bodies that fell on them can be evaluated from these inclinations. An estimate performed with allowance for the size distribution function of bodies (Safronov, 1965a) revealed that the masses of the largest bodies settling on the Earth amounted to about 10^{-3} times the Earth's mass. From Section 26 it is evident that this mass ratio is related by expression (9.7) to the bodies' relative velocities and that it makes it possible to evaluate the parameter θ characterizing these velocities.

Knowledge of the sizes of the largest bodies that fell on the Earth is also important for geophysics: it is required for the determination of the Earth's initial temperature (see Chapter 15) and makes it possible to estimate the scale of primary inhomogeneities of the Earth's mantle (see Chapter 16).

In Chapter 10 we remarked that the observed rotation of the planets breaks down into two components: a systematic (regular) component with momentum $\mathbf{K}_1$ at right angles to the central plane of the planetary system (direct rotation) and a random component $\mathbf{K}_2$ manifested in the inclination of the planets' axes of rotation. The latter is related to the discreteness of the process of planetary growth. It shows that a considerable fraction of the mass settled on the planet in the form of individual bodies with randomly oriented relative motion at the instant of impact. A characteristic feature of the planetary system is that the angles of inclination of the axes of most of the planets are of the same order of magnitude. It points to a definite pattern of growth, to a pattern governing the size distribution of the bodies.

Let m and r be the mass and radius respectively of a growing planet, and let m_i' be the mass of a body falling on it. For clarity we will begin with the case where all falling bodies have the same masses $m_i' = m'$ and move in the plane Oxy with reference to the planet m whose center lies at the point O. Let v be the velocity of a body with reference to the planet before impact. Then

the angular momentum imparted to the planet by the mass m'_i,

$$\Delta K_{2i} = m'vl_i \tag{1}$$

is directed along the z-axis and is a random variable, since the impact parameter l_i of the incident body is a random variable with constant probability density in the interval $(-l_0, +l_0)$. The expectation value of l (mean value of l) is zero, but the expectation value of l^2 (variance of l) is nonzero:

$$\mathbf{M}l = \bar{l} = 0, \qquad \mathbf{D}l = \overline{l^2} = \frac{1}{2l}\int_{-l_0}^{+l_0} l^2 dl = \frac{1}{3} l_0^2. \tag{2}$$

The quantity l_0, the largest impact parameter leading to collision between m' and m, is related to the radii r and r' by relation (9)

$$l_0^2 v^2 = (r + r')^2 \left[v^2 + \frac{2G(m + m')}{r + r'}\right], \tag{3}$$

which is an elementary consequence of the laws of conservation of energy and angular momentum in a two-body system.

For $m'v = \text{const}$, when several bodies m' fall on the body m we have, from the theorem for the addition of variances of a sum of independent random variables (Gnedenko, 1962),

$$\mathbf{D}\sum_{i=1}^{n} \Delta K_{2i} = (m'v)^2\, \mathbf{D}\sum_{i=1}^{n} l_i = (m'v)^2 \sum_{i=1}^{n} \mathbf{D}l_i = (m'v)^2 \frac{nl_0^2}{3}. \tag{4}$$

Consequently, the mean value of the square of the angular momentum imparted by bodies m' with total mass $\Delta m = nm'$ is given by

$$\overline{\Delta K_2^2} = (m'v)^2 \frac{nl_0^2}{3} = (vl_0)^2 \frac{m'}{3} \Delta m. \tag{5}$$

The magnitude of the random component ΔK_2 of the angular momentum imparted to the planet by incident bodies is obviously determined by its mean square value, which is related to m' by expression (5). From (5) it is obvious that the angular momentum imparted increases with the size of the body m'. Small particles contribute practically nothing to ΔK_2.

In the more general case of bodies moving in all possible directions the random component of the imparted angular momentum can be evaluated as follows. Let one third of all bodies $(n/3)$ move parallel to the x-axis, one third parallel to the y-axis, and one third parallel to the z-axis. This simplification is frequently employed in the kinetic theory of gases.

Consider the bodies moving along the z-axis before encountering the planet. When they fall onto the planet they will impart to it the angular momentum components K_{2ix} and K_{2iy} along the x- and y-axes respectively. Obviously,

$$K_{2ix} = m'vl \sin\varphi, \qquad K_{2iy} = m'vl \cos\varphi, \tag{6}$$

where φ is the angle between the plane Oxz and the plane of the body's orbit

with reference to the planet. The variance of the random variable K_{2ix} is given by

$$\mathbf{D}K_{2ix} = \overline{K}^2_{2ix} = (m'v)^2 \frac{\int_0^{l_0}\int_0^{2\pi} (l \sin\varphi)^2\, l\,dl\,d\varphi}{\int_0^{l_0}\int_0^{2\pi} l\,dl\,d\varphi} = \frac{(m'vl_0)^2}{4}. \tag{7}$$

Similarly, $\mathbf{D}K_{2iy} = \frac{(m'vl_0)^2}{4}$.

An angular momentum component along the x-axis will also be contributed by bodies moving parallel to the y-axis before encountering the planet; the variance $\mathbf{D}K_{2ix}$ is given by the same expression (7). The variance of the sum of random variables K_{2ix} is equal to the sum of the variances of the terms

$$\mathbf{D}\sum_{i=1}^{n} K_{2ix} = 2\,\frac{n}{3}\,\mathbf{D}K_{2ix} = \frac{n}{6}\,(m'vl_0)^2. \tag{8}$$

The angular momentum components along the y- and z-axes will have the same variance. According to (8), the expectation value of the square of the angular momentum component along the x-axis is

$$\Delta K^2_{2x} = \frac{v^2 l_0^2 m' \Delta m}{6}. \tag{9}$$

Consequently,

$$\Delta K^2_2 = \Delta K^2_{2x} + \Delta K^2_{2y} + \Delta K^2_{2z} = \frac{1}{2}\, v^2 l_0^2 m' \Delta m. \tag{10}$$

We introduce $v^2 l_0^2$ from (3) above, taking $v^2 = Gm/\theta r$ in the right-hand side in accordance with (7.12) (θ is of the order of a few units) and dropping the terms m' and r', which, as will be evident from what follows, are small compared with m and r. Then

$$\Delta K^2_2 = \left(1 + \frac{1}{2\theta}\right) Gmrm'\Delta m = \left(1 + \frac{1}{2\theta}\right) Gmrnm'^2. \tag{11}$$

The specific angular momentum imparted is inversely proportional to the square root of n:

$$\Delta K_2/\Delta m = \sqrt{(1 + 1/2\theta)\, Gmr/n}. \tag{11'}$$

From the rule of summation of variances it is easy to obtain an expression for ΔK^2_2 in the more general case where the masses m'_j of the falling bodies vary. To do this expression (11) must be summed over all m_j. Let $n(m')$ be the mass distribution of bodies incident on the planet and having total mass

$$\Delta m = \int_0^{m_0} m' n(m')\, dm'. \tag{12}$$

Integrating (11) over all m' and introducing Δm from (12), we obtain

$$\Delta K_2^2 \approx \left(1+\frac{1}{2\theta}\right) Gmr \frac{\int_0^{m_1} n(m')\, m'^2 dm'}{\int_0^{m_1} n(m')\, m' dm'} \Delta m, \tag{13}$$

where m_1 is the mass of the largest body, not counting the planet itself. This relation is obviously meaningful only for $m_1 \ll \Delta m \ll m$.

The expression

$$J(m, m_1) = \frac{\int_0^{m_1} n(m')\, m'^2 dm'}{m \int_0^{m_1} n(m')\, m' dm'} \tag{14}$$

is a function of the planet mass m, since $n(m')$ depends on time. If

$$n(m', t) = c(t)\, m'^{-q}, \tag{15}$$

then for $q < 2$

$$J(m, m_1) = \frac{\int_0^{m_1} m'^{2-q} dm'}{m \int_0^{m_1} m'^{1-q} dm'} = \frac{2-q}{3-q} \frac{m_1}{m}. \tag{16}$$

The masses of the falling bodies m' increase with the planet's growth, and therefore m_1/m can be regarded as constant in the first approximation. Then $J =$ const if $q =$ const. Regarding the planet's density as constant and integrating (13) over m, we obtain the square of the random component of rotational momentum of the planet

$$K_2^2 = \sum \Delta K_2^2 = \left(1+\frac{1}{2\theta}\right) GJ \int_0^m m^2 r dm \approx \left(1+\frac{1}{2\theta}\right) GJ \cdot \frac{3}{10} m^3 r$$

and thus

$$K_2 = m \sqrt{\frac{3}{10}\left(1+\frac{1}{2\theta}\right) JGmr}. \tag{17}$$

Allowance for increasing planet density with m scarcely affects the result; the right-hand side of (17) increases only by a quantity of the order of 1% in all.

By definition the vector $\mathbf{K}_2$ has a random orientation in space. Let the angle between the systematic angular momentum component $\mathbf{K}_1$ perpendicular to the orbital plane and $\mathbf{K}_2$ be ψ, and the angle between $\mathbf{K}_1$ and the total angular momentum vector of the planet $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2$ (inclination of the axis of rotation) be ε. Then the angular momentum component perpendicular to z is given by

$$K_2 \sin\psi = K \sin\varepsilon. \tag{18}$$

The right-hand side of the above is known from observations. In the left-hand side K_2 is given by relation (17) and the angle ψ can take any value between 0 and π. For the probable value of $\sin\psi$ in (18) it is natural to take its mean value. For uniform distribution of the vectors $\mathbf{K}_2$ over the sphere

$$\overline{\sin^2\psi} = \frac{1}{4\pi}\int_0^\pi \sin^2\psi 2\pi \sin\psi d\psi = \frac{2}{3}. \tag{19}$$

Introducing $\overline{\sin^2\psi}$ and K_2 expressed in terms of m_1/m with the aid of (16) and (17) into (18), we obtain the expectation value of m_1/m:

$$\frac{m_1}{m} = \frac{3-q}{2-q}\frac{5\sin^2\varepsilon}{(1+1/2\theta)}\frac{K^2}{Gm^3r}. \tag{20}$$

For numerical computations it is convenient to introduce the velocity of rotation v_r of the planet at the equator and the circular Kepler velocity v_e at the planet's surface:

$$K = \frac{2}{5}\mu m r v_r, \qquad v_e = \sqrt{Gm/r}. \tag{21}$$

Then from (20) we obtain

$$\frac{m_1}{m} = \frac{3-q}{2-q}\frac{4}{5(1+1/2\theta)}\left(\mu\sin\varepsilon\frac{v_r}{v_e}\right)^2. \tag{22}$$

The masses of the largest bodies falling on the planet as calculated from this formula for a power distribution function with $q = 5/3$ (distribution over radii with exponent $p = 3q - 2 = 3$) are given in the first column of Table 12 and in Figure 8.

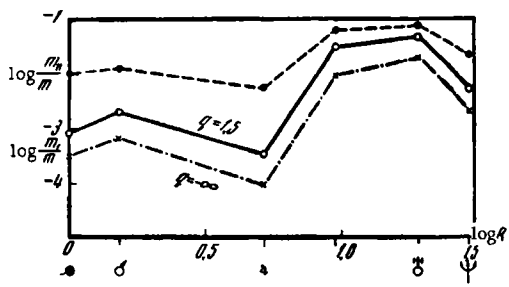

FIGURE 8. Masses of largest bodies m_1 falling on planets during their period of formation, evaluated from the inclination of the planets' axes of rotation. The unit is the planet mass m.

Lower curve — all incident bodies were of the same size. Central curve — incident bodies had power law of mass distribution with exponent $q = 1.5$. Upper curve — inclination of axis of rotation due to fall of a single body of mass m_{11}.

For Uranus $\sin\epsilon$ was replaced by the ratio $\pi K_2/4K$, which was obtained under the assumption that the systematic angular momentum component K_1 of Uranus corresponds to a period of rotation of 15 hrs (approximately the same as for Neptune).

TABLE 12

Planet	m_1/m		m_{11}/m
	$q = 5/3$	$q = -\infty$	
Earth	$1 \cdot 10^{-3}$	$3 \cdot 10^{-4}$	$1 \cdot 10^{-2}$
Mars	$2 \cdot 10^{-3}$	$6 \cdot 10^{-4}$	$1.3 \cdot 10^{-2}$
Jupiter	$3 \cdot 10^{-4}$	$9 \cdot 10^{-5}$	$5 \cdot 10^{-3}$
Saturn	$4 \cdot 10^{-2}$	$1 \cdot 10^{-2}$	$6 \cdot 10^{-2}$
Uranus	$7 \cdot 10^{-2}$	$2 \cdot 10^{-2}$	$8 \cdot 10^{-2}$
Neptune	$7 \cdot 10^{-3}$	$2 \cdot 10^{-3}$	$2 \cdot 10^{-2}$

For $\theta \geqslant 3$ the role of the parameter characterizing the relative velocities of bodies before approach to the planet in (22) is insignificant. We have assumed $\theta = 3$. For $q = 1.8$ the values obtained for m_1/m are twice as large as for $q = 5/3$.

For $q = 2$ (i. e., $p = 4$), $(3-q)/(2-q)$ in the right-hand side of (22) should be replaced by $\ln(m_1/m_{min})$, where m_{min} is the mass of the smallest particles in the given distribution. Here the values of m_1/m become 2—3 times as large as for $q = 5/3$. In Section 25 it was shown (see Chapter 8) that q lies in the interval $3/2 < q < 2$ and that it is probable that it does not depart considerably from the value $q = 1.8$.

Computations were carried out under the assumption that $m_1/m =$ const. Variation of m_1/m affects the results only weakly: for $m_1 \propto m^2$ the values of m_1/m would be 30% higher than given in Table 12 for $m_1/m =$ const, while for $m_1 =$ const its values would be 30% lower.

The second column of the table lists the values of m_1/m for $q = -\infty$, i. e., for the case where all falling bodies are of the same mass. These values are three times smaller than the preceding set. One could consider yet another extreme case where the random angular momentum component K_2 is imparted only by a single body m_{11} while all the remaining matter falling on the planet imparts to it only a regular rotation (K_1). Then

$$K \sin \epsilon = \frac{\pi}{4} K_2 = \frac{\pi}{4} m_{11} l v = \frac{\pi}{4} m_{11} \frac{2}{3} l_0 v \tag{23}$$

and the expectation value of m_{11}/m is given by

$$\frac{m_{11}}{m} = \frac{6\sqrt{2}\mu}{5\pi\sqrt{1 + 1/2\theta}} \frac{v_r}{v_e} \sin \epsilon. \tag{24}$$

The values of m_{11}/m are given in the last column of the table. They vary less from planet to planet than do the values of m_1/m. The values of m_{11}/m can be regarded as the upper limit for the masses of bodies falling on the planet.

We see from these results that despite the absence of definitive data on the size distribution function for the bodies, the masses of the largest bodies falling on the planet in the process of their formation can be determined with relative certainty, with no more than threefold deviation in either direction. The masses of the largest bodies incident on the Earth amounted to about 10^{-3} Earth masses. Due to the tidal effect of the Moon the Earth's rotation is slowing down, and although the axial inclination ϵ is increasing, the

quantity $v_r \sin\epsilon$ is decreasing (Gerstenkorn, 1955). If one assumes that the Moon's original distance was half the present one, the value of m_1/m obtained above for the Earth should be increased slightly less than twice.

The retrograde rotation of Uranus can be explained naturally by the relatively larger sizes of the bodies from which it was formed. The masses of the largest bodies falling on Uranus reached 0.07 planet masses. The bodies in Saturn's zone of formation were also of considerable size. The largest of these amounted to 0.04 planet masses. Consequently, with regard to rotational anomaly, Saturn differs only slightly from Uranus. The causes of the anomalies are related to the influx into the zones of these planets of larger bodies from the Jupiter zone. In Section 31 it will be shown that Jupiter grew much more rapidly and that it began scattering bodies earlier by virtue of its own gravitational disturbances into the zones of other planets. It should be mentioned that the estimates of m_1/m cited above for Jupiter and Saturn require substantial revision as they fail to account for the accretion of gaseous hydrogen in the closing phases of growth of these planets. However, such revision will be possible only when a theory of growth has been developed for these planets.

Chapter 12

GROWTH OF THE GIANT PLANETS

31. Duration of growth process among the giant planets

The growth of the giant planets was complicated by a number of important factors, including first and foremost fusion of source zones, ejection of bodies beyond the solar system by gravitational disturbances, dissipation of gas away from the giant planet region, and the accretion of hydrogen by Jupiter and Saturn.

Evaluation of the planets' rate of growth indicates that for the outermost planets (Uranus and Neptune) formulas like (9.18) yield an inadmissibly long growth span — 10^{11} years (Safronov, 1954). To circumvent this difficulty one would have to assume either considerably lower relative velocities for the bodies or a considerably larger mass of material inside this region (Safronov, 1958a).

Table 13 gives the characteristic time τ_0 of exhaustion of available matter by the giant planets, as calculated from formula (9.16) for their present masses and densities under the assumption that the planetary zones were isolated. It would appear from the table that the distant planets (Uranus, Neptune and Pluto) could not have managed to develop and use up all the matter in their zones within the lifetime of the solar system. For Neptune τ_0 is 10^2 times greater than the maximum admissible value.

TABLE 13

	Jupiter		Saturn		Uranus		Neptune		Pluto	
σ_0	95		5.7		0.3		0.25		10^{-3}	
θ	3	5	3	5	3	5	3	5	3	5
τ_0, billion years	0.055	0.035	1.02	0.65	47	30	148	94	417	265
m_g (Earth masses)	0.12	0.26								
m_U	0.70	1.5	0.07	0.15						
m_N	1.30	2.7	0.23	0.50	0.009	0.02				
m_P	1.6	3.5	0.35	0.75	0.03	0.06	0.001	0.002		
m_∞	3.2	6.8	1.3	2.8	0.45	0.99	0.23	0.50	0.15	0.33

In the initial stage of growth, when the masses of these planets were still small, their zones did not overlap. The growth formula (9.18) enables us to obtain the growth times of the planet embryos at this stage if the initial surface density σ_0 of matter in their zones is known. Some idea of the

duration of the initial stage of growth can be obtained by examining the next rows of the table. They give the planet masses for which bodies having the corresponding relative velocities ($v = \sqrt{Gm/\theta r}$, for $\theta = 3$ and $\theta = 5$) will be at aphelion along their orbits at the distances of the other planets (indicated by the subscripts on m) if v is directed along the orbit and forward (with reference to the planet's motion). The masses are given in terms of the Earth's mass. Thus the mass of Jupiter for which bodies could have traveled from its zone to the distance of Saturn is given in the column headed Jupiter and in the row designated by m_s. For $\theta = 3$ and 5 it is 0.12 and 0.26 Earth masses. Since the inner planets grew more rapidly than the outer ones, the zones of Jupiter and Saturn were the first to fuse. At this stage $m \ll Q$, and one can use the simplified growth formula (9.21). If the accretion of hydrogen by Jupiter had not yet begun at this time, the σ_0 appearing in τ_0 should refer to solid matter alone. The mass taken in Section 32 for the protoplanetary cloud ($0.05\ M_\odot$) corresponds to $\sigma_0 \approx 20\,\mathrm{g/cm^2}$. Bodies from the Jupiter zone should then have flown out to the distance of Saturn within 50–60 million years after Jupiter had first begun to form, and to the distance of Neptune within 100–130 million years; they would have escaped beyond the solar system within 150–170 million years. The escape of bodies from Saturn's zone began considerably later – 10^9 years to the distance of Uranus.

Thus already 150 million years after the large planets had begun to grow, bodies from the zone of Jupiter were shooting through the entire outer portion of the solar system as well as through the zones of the asteroids and Mars. Since the zones of these planets ceased to be isolated, the foregoing formula of planetary growth becomes inapplicable. The growth process was further complicated by the presence of gases. At first the role of the gases was confined to lowering the relative velocities of particles and bodies by retardation. As we saw in Section 22, the value of θ may have been substantially higher. Since the masses of planetary embryos permitting escape of bodies (given in Table 13) are proportional to $\theta^{3/2}$ and the rate of growth is proportional to $1 + 2\theta$, in the case of Jupiter (for example) for $\theta = 30$, $m_s = 3.9$ and the time required for growth to this mass is about 60 million years. For this value of the mass accretion of hydrogen will already have taken place, leading to considerably faster growth of the planetary embryo due to the fact that the radius r_a of capture by accretion is proportional not to the radius of the embryo but to its mass:

$$r_a = \sqrt{2\alpha} = \frac{Gm}{v^2 + c^2}, \tag{1}$$

where $1 < \alpha < 2$; v is the velocity of the gas with reference to the planet embryo and c is the thermal velocity of the molecules (Bondi and Hoyle, 1944; Bondi, 1952).

When the mass of the embryo is sufficiently large, the energy imparted to it by the gas leads to considerable warming of its surface. An approximate computation for $v^2 + c^2 = 1\,\mathrm{km/sec}$ (Safronov, 1954) indicated that the maximum temperature reached for mass equal to 2/3 of the contemporary planet mass was 17,000° for Jupiter and 3600° for Saturn. The result depends on the estimated gas density, which is unreliable in view of the fact that the initial mass of the cloud and the rate of dissipation of the gas are unknown. But since the temperature is determined from the balance of absorbed and emitted energy and is proportional to the fourth root of the gas

density, the estimate should be correct as to order of magnitude. The higher densities of satellites adjacent to Jupiter are probably attributable to the high temperature of Jupiter during their formation.

The growth times of Uranus and Neptune can be determined approximately from expression (9.21) for the growth of bodies in a medium of constant density, which can be written as follows:

$$t = \frac{P\delta r}{(1+2\theta)\sigma} \approx \frac{P\delta r}{2\theta\sigma}. \qquad (2)$$

From here it is easy to find the value of $\bar{\sigma\theta}$ required for the planet to complete its growth within a time not exceeding 4 billion years. Then for Uranus $\bar{\sigma\theta} > 40$ and for Neptune $\bar{\sigma\theta} > 80$. Since $\bar{\sigma} < \sigma_0$, one needs $\sigma_0\theta \approx 10^2$. The high values of $\sigma_0\theta$ in the outer-planet region lead one naturally to conclude that the ejection of bodies from the solar system played a considerable part in the process of formation of these planets.

32. Ejection of bodies from the solar system

The ejection of matter from the solar system is already mentioned by Oort (1950, 1951) in his theory of the origin of comets. Oort conjectured that the comets were formed together with the planets in a single process, and were ejected by Jupiter's perturbations from the asteroid zone beyond the confines of the solar system. About 5% of the total number of bodies ejected continued to travel around the Sun at large distances under the influence of perturbations from stars closest to the Sun within the so-called comet cloud. There they were effectively preserved owing to the low temperature. They occasionally reenter the planetary system under the influence of renewed stellar perturbations, becoming observable as comets as they draw near the Sun. Levin (1960), having noticed that in the closing phases of growth the relative velocities of bodies in the giant planet region (which are proportional to the parabolic velocity at the surface of the planet) exceeded the parabolic velocity at the distances of these planets from the Sun, concluded that the ejected mass may have been substantial and that the masses of the giant planets do not determine the initial mass of matter in this region; they represent instead a kind of limiting mass. Having achieved this mass the planet practically ceases to grow further, since in the main it ejects the bodies drawing close to it and fails to use them up.

If the mass of solid matter ejected was considerable, the total initial mass of the protoplanetary cloud must have been correspondingly greater. The lower limit of the cloud mass is usually evaluated by adding volatile substances to the matter of the planets until the solar composition is reached. According to Whipple (1964), to reach the solar composition the mass of the planets in the Earth group must be increased 500 times; the mass of Jupiter must be increased 10 times, that of Saturn 30 times, that of Uranus and Neptune 75 times. This gives a minimum initial mass for the cloud of $0.028\ M_\odot$.

Whipple gives $0.003\ M_\odot$ for the mass of ejected substance and $0.031\ M_\odot$ for the total mass of the cloud. Data for Jupiter and Saturn are not in contradiction with computations of the hydrogen content in these planets carried

out by Kozlovskaya (1956). However, these results were based on models of the giant planets not containing helium. The addition of helium reduces the content of heavy elements and brings the composition closer to the solar one. Kieffer (1967) has recently constructed a model of Jupiter using material of solar composition and one of Saturn using material roughly 5% more dense. The lower limit of the cloud mass for this particular composition of these planets decreases by a factor of 2—3. However, the estimated compositions of Jupiter and Saturn remain unreliable in view of the absence of a reliable equation of state for hydrogen and helium mixtures.

A "ceiling" estimate of the initial mass of the cloud can be carried out independently for the gaseous component and for the solid matter (Safronov, 1966a).

1. All the planets (with the exception of Jupiter and Saturn) differ substantially in chemical composition from the Sun, and none could have formed as a result of gravitational instability in the gaseous component of a cloud whose composition is assumed to be solar. The mechanism of thermal dissipation could not have produced practically total sorting and removal of hydrogen and helium (originally amounting to 98% by mass) from massive self-gravitating condensations (Shklovskii, 1951). With formulas (6.6) and (5.17), it can be shown that condensations, formed in the giant planet region as a result of gravitational instability inside the gases, should have had masses equaling about 15 Earth masses. It seems one must rule out the possibility of hydrogen and helium separating out of such massive bodies under the conditions prevailing in the protoplanetary cloud.

It cannot be concluded from the chemical composition of Jupiter and Saturn that gravitational instability could not have been maintained in the gases of their zones. However, the assumption of such instability raises considerable difficulties. For instability to have appeared in the gas the gas density must not have fallen below the critical value $2.1\,\rho^*$ (see Section 16) and the total mass must not have been less than 60 Jovian masses in each zone. Instability could have spread over the entire planetary zone, leading to the formation of about 10^3 condensations. If, on the other hand, instability affected a small part of the zone, the number of condensations may have been small. However, in neither case is it clear why the process of interaction and fusion of the condensations into a developing planet involved only 1% of the matter inside the zone while 99% was ejected from the solar system. Below it will be shown that the mass of bodies ejected by the planet in encounters between it and the bodies is merely one order greater than the mass of bodies falling on the planet. This could induce one to believe that gravitational instability was absent in the gases of the Jupiter and Saturn zones.

Thus the gas density ρ_0 in the central plane of the cloud need not have reached the critical density $2.1\rho^*$. A reasonable upper limit for ρ_0 could be the value ρ^*, corresponding to the total mass of the cloud in the zone of the large planets, namely $0.12\,M_\odot$. It has also been mentioned that it is necessary to bound the cloud mass from the top by the value $\sim 0.1\,M_\odot$, which is determined by the possibility of dissipation of this entire mass of gas from the solar system (Kuiper, 1953; Ruskol, 1960).

2. A large initial cloud mass implies dissipation of a large quantity of solid (not just gaseous) material from the solar system. The mechanism of ejection of bodies by gravitational perturbations appears to be efficient, but it entails grave consequences related to the redistribution of angular

momentum. An ejected body must raise its absolute velocity to the parabolic velocity. When a body encounters the planet, its relative velocity vector rotates without changing magnitude. Its absolute velocity will increase if this rotation takes place along the direction of orbital motion of the planet (whose orbit can be regarded as circular). In the process the angular momentum of the body with reference to the Sun will increase due to the orbital momentum of the planet. Consequently bodies are predominantly ejected in the direction of the planet's motion. If the total mass of the ejected bodies is comparable to the mass of the planet, the planet will draw noticeably closer to the Sun. This might account for Neptune's violation of Bode's law: for the distance from Neptune to the Sun to decrease from 40 to the present-day 30 a. u., it is sufficient that Neptune should have ejected from the solar system a mass of bodies equal to one third of its own mass.

The condition of angular momentum conservation gives the relation between the mass dm_{ej} ejected from the solar system from a distance R in the direction of rotation and the Sunward displacement $-dR$ of the remaining mass m:

$$(\sqrt{2}-1)\sqrt{GM_{\odot}R}\,dm_{ej}=-md\sqrt{GM_{\odot}R}. \qquad (3)$$

Ejection of the mass m_{ej} causes the distance R of the mass m from the Sun to reduce to R', as given by

$$m_{ej}=1.21\ln\left(\frac{R}{R'}\right)m. \qquad (4)$$

For $R/R'={}^4/_3$, $m_{ej}\approx m/3$ and for $R/R'={}^3/_2$, $m_{ej}\approx m/2$. Assuming that the mass of solid material in the giant planets equaled 50 Earth masses and that the mass of solid material in a cloud of solar composition would equal 1/75 of the cloud mass, we find that in the first case ($R/R'={}^4/_3$) the mass of the protoplanetary cloud is $0.05\,M_{\odot}$ and in the second case it is $0.06\,M_{\odot}$. The larger values of R/R' seem implausible since the small mass of Mars and the arrestation of the process of fusion in the asteroid zone show that Jupiter's distance from the Sun during its formative period did not substantially exceed its present distance. It would also be difficult to accept the larger values of m_{ej}, since the amount of solid material ejected by the planets should not have been far in excess of the amount included in the planets. Therefore $0.05-0.06\,M_{\odot}$ seems to us a reasonable upper limit for the initial mass of the protoplanetary cloud.

When the Jovian embryo had grown to the size of 2–3 Earth masses it scattered a considerable portion of the material in its zone over the entire large planet region. The addition of 100–200 Earth masses in the Uranus and Neptune zones would make it possible to raise the surface density σ of solid matter in these zones by one order. For Neptune to grow at the required rate, one must further increase the efficiency of collisions between the bodies and Neptune by one order, i. e., increase the gravitational focusing while reducing the bodies' relative velocities (increase θ). At the initial stage of growth θ may have been appreciably larger owing to damping of the bodies by the gases (Section 22, Table 7). But by the time Neptune reached a mass of terrestrial order the gases had largely escaped from its zone: otherwise there would have been accretion of gas by the planet (see Section 33)

and Neptune would have contained appreciable amounts of free hydrogen. By the time Neptune reached a mass such that it could eject bodies in its zone beyond the confines of the solar system, relative velocities of bodies in its zone practically ceased to increase further, since all bodies whose absolute velocities V grew to the parabolic velocity V_e left the system. If we designate by v_{ej} the mean relative velocity at which ejection took place, we have $v_{ej}=\sqrt{Gm/\theta_{ej}r}$. Further increase in the planet mass was therefore accompanied by an increase in θ_{ej}.

Ejection of bodies proceeded predominantly in the direction of rotation of the system. The velocity vector $\mathbf{v}_{ej}$ of the ejected body deviated from the tangent to the circular planet orbit by an angle not exceeding φ, related to v_{ej} by

$$V^2=V_e^2+v_{ej}^2+2V_e v_{ej}\cos\varphi=V_e^2=2V_e^2,$$

whence

$$v_{ej}=V_e(\sqrt{1+\cos^2\varphi}-\cos\varphi)=V_e u(\varphi). \qquad (5)$$

The angle φ was determined by the efficiency of the mechanism of growth of the bodies' relative velocities. It increased as the planet mass grew, amounting to about 45° in the last stage. Table 14 lists values of v_{ej}/V_e as a function of φ, as well as corresponding values of θ_{ej} computed for present-day masses and radii of the giant planets and typical, therefore, of the concluding stage of growth.

TABLE 14

	φ, °				
	0	15	30	45	60
v_{ej}/V_e . . .	0.414	0.424	0.456	0.516	0.62
θ_J	63	60	52	41	28
θ_S	40	38	33	26	18
θ_U	31	30	25	20	14
θ_N	60	58	49	39	27

The parameter θ characterizing the mean relative velocity of all bodies in the planetary zone is obviously greater than θ_{ej}:

$$\theta=\theta_{ej}v_{ej}^3/\overline{v^3}.$$

In the closing stages of growth of the planetary giants for a Maxwellian velocity distribution $\theta\approx(2-3)\,\theta_{ej}$.

Thus θ does appear to have been of the required order of magnitude in the last stage of growth. But it is unclear what θ was in the intermediate stage. Bodies collided infrequently owing to the low density, and "foreign" bodies (impinging from other zones) had higher velocities. No one has analyzed

the accumulation process in this region while allowing for all essential factors, and as yet there is no theory of growth of the giant planets.

It is still not clear how much solid material was ejected from the solar system in the course of the giant planets' growth. Oort and Whipple suppose that its mass was one order greater than the present mass of the comet cloud, i. e., about 1—10 Earth masses. The inference that the ejected mass was small is drawn in an interesting work by Öpik (1965) devoted to dynamical aspects of comet formation. It is based on an analysis of the mechanism by which bodies speed up during encounters with a planet. Öpik concludes that only large bodies of the order of comet nuclei (1 to 100 km in diameter) could have been ejected over large distances. Smaller bodies were braked in collisions with other bodies and remained inside the system. But Öpik presupposes an excessively high density of matter in the system — $\rho \approx 10^{-10} j$, where j is the fraction of the planet mass that had not been used up. If the fraction j of the present mass of Jupiter were to be scattered over the entire large planet zone up to the distance of Neptune, the mean density of matter in this volume (equal to 0.4—0.5 for contraction along z) would be $10^{-14} j$, i. e., four orders of magnitude smaller. The lower limit obtained by Öpik for the sizes of the ejected bodies must therefore be decreased from 1 km to 10 cm. Moreover, Öpik assumes that bodies encountering the planet increase their relative (random) velocities only in the case of elliptical planetary orbits. For small eccentricities of the planetary orbit the velocity dispersion of the bodies increased slowly. This apparently holds when the variation of the bodies' orbits is affected by perturbations from one planet alone. Owing to the low density encounters among bodies were rare, and the resulting mutual perturbations small (although they should have been substantial for the density assumed by Öpik). But the proximity of other planets or their embryos (see Section 26) would cause the velocity dispersion of the bodies to increase even if planetary orbits were circular (the mechanism of velocity growth being similar to that described at the end of Section 23).

It follows from these remarks that the mass ejected by the planets could have been appreciably larger than Öpik suggests. We saw earlier that unless we increase the mass of the ejected material accordingly, the growth times of the distant planets would involve us in considerable difficulties.

One can make a rough estimate of the amount of material ejected using the results of Chapter 7. Encounters between bodies and a planet are characterized by the relaxation time T_D^* and encounters among bodies themselves by the relaxation time T_D''. Within the time τ_g, the inverse of which is equal to the sum of the inverses of T_D^* and T_D'' (see 7.84), the body's relative velocity vector $\mathbf{v}$ turns as a result of encounters through an angle $\pi/2$ on the average. Within the time τ_{g*} (which differs from τ_g in that, if $T_D^* < T_D''$, it will contain T_D^* instead of T_D'') the rotation of $\mathbf{v}$ causes bodies to acquire an energy of relative motion amounting on the average to the following quantity per unit mass (according to (7.29)):

$$\tau_{g*}\varepsilon_1 = \beta'' v^2. \qquad (6)$$

At the closing stage of growth $\tau_{g*} > \tau_g$ and the number of turns of the vector $\mathbf{v}$ involving an increase in v is substantially less than the total number of turns. The turns are mainly due to distant encounters, which are one order more efficient than close encounters. Body velocities thus increase gradually to $v_{ej} = V_* u(\varphi)$ (see (5)). Further growth ceases since bodies reaching this

velocity leave the solar system as soon as the vector $\mathbf{v}$ enters a cone of aperture φ and axis coincident with the direction of motion of the planet around the Sun.

Let $d\mu_{ej}$ be the fraction of bodies ejected within time dt. The equation of conservation of energy per unit mass can be written as

$$dv^2 + (v_{ej}^2 - v^2)\, d\mu_{ej} = 2(\varepsilon_1 - \varepsilon_2)\, dt, \tag{7}$$

where v_{ej}^2 is the mean square relative velocity of the ejected bodies and $\varepsilon_2 = \zeta' v^2/\tau_e''$ is the energy of relative motion lost in collisions among the bodies. We denote by λ the ratio of the frequency of ejections of bodies from the solar system to the frequency of incidences on the planet. Then the lifetime of a body before ejection is given by τ_e^*/λ, where τ_e^* is the lifetime before incidence on the planet. Since

$$dv^2 = d\left(\frac{Gm}{\theta r}\right) = \left(\frac{2}{3}\frac{dm}{m} - \frac{d\theta}{\theta}\right) v^2 = \frac{2}{3} v^2 \frac{Q-m}{\lambda m}\, d\mu_{ej} - v^2 \frac{d\theta}{\theta},$$

we have

$$d\mu_{ej} = \frac{2(\beta''/\tau_{ge})(1 - \varepsilon_2/\varepsilon_1)\, dt + d\theta/\theta}{v_{ej}^2 - v^2[1 - 2(Q-m)/3m\lambda]}\, v^2. \tag{8}$$

In the closing stage of giant planet growth $\lambda \gg 1$ and $2(Q-m)/3m\lambda \ll 1$. The second term in the numerator $(d\theta/\theta)$ is also small and can be disregarded. Moreover, the expended energy ε_2 is several times smaller than ε_1. The fraction of bodies landing on the planet in time dt is given by dt/τ_e^*. Therefore

$$\lambda = \tau_e^* \frac{d\mu_{ej}}{dt} \leqslant \frac{2\beta'' v^2}{v_{ej}^2 - v^2} \frac{\tau_e^*}{\tau_{ge}}. \tag{9}$$

Assuming $T_D^* = \chi T_D^*$, $\chi > 1$ and $\tau_{ge} = \chi T_D^*/2$, we obtain, from (7.81) and (7.82),

$$\lambda \leqslant \frac{16\beta'' f_3 v^2}{\chi(v_{ej}^2 - v^2)} \frac{\theta^2}{1 + 2\theta} \approx \frac{4\beta'' f_3}{\chi} \frac{v_e^2}{v_{ej}^2 - v^2}, \tag{10}$$

where $v_e = \sqrt{2Gm/r} = v\sqrt{2\theta}$ is the parabolic velocity at the planet surface. According to (7.79), $f_3 \approx 2\ln(D_M/\theta r) - 1$. The expressions for the relaxation times were obtained from computations of the angle of deflection in encounters carried out within the two-body problem. Therefore the maximum distance D_M to which the overall result of distant encounters is reckoned should not exceed the radius of the sphere of action or the distance to the libration point L_1 of the planet m. For Jupiter $r_{L_1} \sim 10^3$. Then $f_3 \approx 4$. The mean square relative velocity v^2 increases with time; for a Maxwellian velocity distribution its upper limit is $(^3/_5)v_{ej}^2$. One can take $v^2 \approx v_{ej}^2/3$. From Table 14 it is clear that for $\varphi < 60°$ the ejection rate is close to $V_e/2$. Lastly, from the results in Section 21 we can take $\beta'' \approx 0.13$. Consequently,

$$\lambda \leqslant \frac{10}{\chi} \frac{v_e^2}{V_e^2}. \tag{11}$$

For Jupiter we obtain $\lambda \approx 200/\chi$. The parameter χ introduces the largest error in the λ estimates. According to (7.87), if in addition to the planet m

there were n bodies of total mass νm, then

$$\chi = \frac{n}{2\nu^2\left(1 + \frac{2\ln n/\nu}{f_3}\right)}. \tag{12}$$

If $n = 10^2$ and $\nu = 1$, then $\chi \approx 15$; if $n = 2$ and $\nu = 1/5$, then $\chi \approx 12$. In Jupiter's last stage of growth it is probable that $\chi > 10$. For $\chi = 15$ we obtain $\lambda \approx 15$. If the composition of Jupiter is similar to the Sun's and the amount of material it contains in solid form amounts to about 10 Earth masses, then for this value of λ the amount of solid matter ejected by Jupiter is equal to half its mass. This estimate agrees with the one obtained earlier from expression (3), which was based on analysis of the angular momentum lost by the planet upon ejection. It is nonetheless very approximate and as yet no final conclusions may be drawn from it.

33. Dissipation of gases from the solar system

Dissipation of gas from the solar system took place in parallel with the evolution of the dust component of the protoplanetary cloud. Since the Earth was growing for $\sim 10^8$ years, the absence on Earth of significant amounts of hydrogen and helium as well as the striking deficit in the noble gases (Brown, 1949) lead one to suppose that these gases escaped from the region of Earth group planets within a period not exceeding $10^{7.5}$ years. Uranus and Neptune also lack significant amounts of helium and free hydrogen. By the time they had become sufficiently massive, the gases had already escaped from their region. The dissipation time probably did not exceed $10^{8.5}$ years. Jupiter and Saturn developed a good deal earlier than Uranus and Neptune. They absorbed all the gas that had not yet escaped from their zones.

It is sometimes assumed that accretion will begin when the body mass is such that the capture radius for accretion exceeds the geometric radius of the body. For a molecular thermal velocity of 1 km/sec this gives $m \geqslant 10^{25}$ g. But bodies of such small mass are unable to retain a hydrogen atmosphere. Accretion theory (Bondi, 1952), which is used to evaluate the rate of gas absorption by gravitating bodies, does not allow for the reflected wave which results when the falling gas strikes against the surface of a body. Thus it cannot tell us for what body mass accretion becomes efficient. If one calculates the atmospheric density for which the quantity of gas acquired by a body by accretion will equal the quantity lost due to thermal dissipation as computed by Jeans' well-known formula, it is found that the mass of this atmosphere will cease to be negligible compared with the body mass only for a body mass of about one to two Earth masses. The result depends on the density and temperature of the gas. In the Saturn zone a body with this mass could have developed in 500–800 million years. The gas should therefore have been preserved in this zone for about 10^9 years.

Kuiper (1953) found that the most efficient mechanism of gas dissipation from the solar system involves knocking out of atoms and molecules at large z-values (in the rarefied portion of the cloud) by high-energy solar corpuscles. The corpuscular flux would have to be large for a considerable mass of gas to dissipate, but it was probably sufficient to ensure dissipation from the region of the Earth group planets. Hoyle (1960) attributes the escape of

gases from the Uranus and Neptune region to thermal dissipation. At the distance of Uranus a particle would dissipate in the direction of the cloud's rotation at a velocity of 3 km/sec with respect to the circular Kepler velocity. Therefore in the Uranus region a gas temperature of 75° is enough to ensure efficient dissipation of hydrogen and a temperature of about 150°K for efficient dissipation of helium. In the presence of a dust layer such temperatures would scarcely be attainable (see Chapter 4) due to rapid cooling of the gas on solid particles. Gold (1963) regards the question of gas dissipation as one of the most difficult ones in planetary cosmogony. In his view the mechanism of thermal dissipation, like every other mechanism involving uniform distribution of energy throughout the cloud, is inefficient. Mechanisms providing concentration of energy in small volumes (solar flares, corpuscular fluxes, perhaps even external interstellar "winds") require appreciably less energy for the dissipation of the same amount of gas.

Schatzman (1967) found that at a distance of 2 a. u. from the Sun, dissipation of the gaseous component of the cloud would have taken place within acceptable time limits if 4% of the energy of the Sun's emission had been ejected during solar flares in the form of high-energy particles and ultraviolet.

In our opinion the most convincing argument against thermal dissipation of considerable masses of gas from the solar system is obtained by analyzing the redistribution of angular momentum (as we did above to evaluate the upper limit on the mass of ejected solid material). Thermal dissipation of gas, like the ejection of solid bodies, takes place preferentially in the direction of rotation of the solar system. The remaining gas loses angular momentum, shifts closer to the Sun, and should be absorbed by Saturn and Jupiter. Thus in the absence of other mechanisms of dissipation, thermal dissipation can account for the loss of only a small mass of gas not exceeding the mass of Jupiter and Saturn, i. e., $\sim 10^{-3} M_{\odot}$, or 2–3% of the required amount. Hoyle assumes a very small cloud mass ($0.01\ M_{\odot}$). Still, this is seven times greater than the mass of the planets. Expression (4) makes it possible to determine the largest mass of gas which the planets could have ejected. Since the actual dissipation of molecules must have taken place at a certain angle φ to the orbit (rather than precisely along it), the factor $u(\varphi)\cos\varphi$ should appear in the left-hand side of (3) instead of $(\sqrt{2}-1)$. But this gives only a slight increase in the numerical coefficient on the right-hand side in (4). For $\varphi = 45°$ it equals 1.37 and for $\varphi = 60°$, 1.55. We then find from expression (4) that even if the initial distance of the planets was 5 times greater than at present, the mass of gas they ejected could not have exceeded the mass of the planets by more than 2–2.4 times. For a cloud mass of $0.05\ M_{\odot}$ the contradiction is even more striking. It is proof of the ineffectiveness of thermal dissipation of gas from the solar system.

Chapter 13

FORMATION OF THE ASTEROIDS

34. Role of Jupiter in the formation of the asteroid belt

Olbers' theory of the formation of the asteroids due to the disintegration of a planet (Phaeton), long popular among astronomers, has been rejected by specialists in recent years. First, the possibility of a planet disintegrating is being seriously disputed from the standpoint of the physics of the disintegration process itself. Second, it has been established that the disintegration of a single planet would not account for the observed distribution of the asteroid orbits, which points to a division of the asteroid system into a series of individual groups (Putilin, 1953; Sultanov, 1953). Third, the meteorites, themselves the products of the fragmentation of asteroids, also fall into groups according to chemical composition (Urey and Craig, 1953; Yavnel', 1956). Fesenkov (1956) notes that the meteorites were formed from bodies of asteroidal size which never combined to form a planet like the Earth, since the crystalline structure characteristic of most meteorites could not have been preserved at great depths in such a planet. Urey (1956, 1958) concluded from physicochemical studies of meteorites that at the time of the meteorites' formation the material from which the planets developed was in the form of solid bodies of asteroidal size.* Shmidt (1954, 1957) stressed that once it has been proved that bodies of asteroidal size could have developed in the course of the planets' growth, there is no further need for special theories regarding the origin of the asteroids. In the asteroid belt the process of planet formation came to a standstill at the intermediate stage of smaller bodies due to the proximity of massive Jupiter, which increased their relative velocities. According to Shmidt, the asteroids' position at the boundary between two groups of planets helped to slow down their growth. Volatile substances originally present in the asteroids subsequently evaporated, reducing their stability and leading to disintegration. In our view the first argument can be accepted in full, the second only in part (see below).

The mean eccentricities and inclinations of the asteroid orbits are $e \approx 0.12$ and $i \approx 12°$ (Putilin, 1953; Piotrowsky, 1953). This corresponds to a relative velocity (relative to the circular Kepler velocity) of 5 km/sec. This velocity dispersion could not have resulted from interaction among present asteroids. It follows from the expression $v = \sqrt{Gm/\theta r}$, discussed in detail in Chapter 7, that relative velocities of 5 km/sec associated with gravitational interaction among bodies in a rotating system are possible only for body masses of the

* A detailed critical review of present-day hypotheses regarding the origin of meteorites was published by Levin (1965).

order of the Earth's mass. It is probable that the observed velocity dispersion of the main asteroid mass is not another result of gravitational perturbations emanating from Jupiter and accumulating throughout the lifetime of the asteroids. A manifestation of these perturbations is the presence of gaps in the region of values of the asteroids' periods of revolution commensurate with Jupiter's period of revolution. These gaps, however, are very narrow, and it seems they are due exclusively to limited variation of the semimajor axis owing to perturbations from Jupiter (and thus to limited variation of the eccentricity). Unfortunately expression (7.12) for v, which is based on analysis of encounters and collisions of random character, cannot be used to evaluate the cumulative effect of the interaction of two bodies moving along nearly circular coplanar orbits.

A more probable conclusion is that the asteroids' velocity dispersion dates back to the era of their formation. Earlier we saw that when the Jovian embryo had become sufficiently massive the bodies in its zone acquired considerable relative velocities and began to scatter into adjacent zones. Upon colliding with bodies in the asteroid zones, they "swept away" most of these, increasing the relative velocities of bodies remaining in the zone. In Section 26 we found that in the zone of action of the largest body the growth of other bodies slows down due to reduced gravitational focusing. The bodies in the asteroid zone were in a similar situation. At first their growth was slowed down, eventually coming to a standstill when the energy of random motion of the bodies had become substantially greater than the potential energy at their surface and collisions among bodies began to lead to fragmentation rather than fusion. Before this stage was reached both fusion and fragmentation took place depending on collision conditions.

At present the kinetic energy of asteroidal bodies exceeds the potential energy at the surface of the largest asteroids by more than one order, and collisions between asteroids end in fragmentation. For a total mass of material in the asteroidal zone amounting to about 10^{-3} terrestrial masses, in 10^9 years every asteroid should experience collisions with other bodies of total mass averaging about 10^{-2} of its own mass. For impacts at a speed of several kilometers per second, this is sufficient to cause substantial destruction of the asteroids.

Thus the fusion of the asteroidal bodies into a single planet was hampered by the proximity of Jupiter's massive embryo, which grew at an appreciably faster rate. If the density of the dust layer had varied smoothly with increasing distance from the Sun, the process could not have displayed such severe irregularity. The asteroid belt lies at the boundary of the region of terrestrial planets, however, and the reason for the anomaly is also the reason why the planets fall into two groups. Condensation of the most abundant volatile elements (CH_4, NH_3 and others) began at Jupiter's distance from the Sun (Levin, 1949). This is evident from temperature conditions in the dust layer (see Chapter 4). The surface density σ of material passing into the solid state was therefore several times higher in the Jovian zone than in the adjacent asteroidal zone. The critical density is proportional to ρ^* and decreases away from the Sun as R^{-3}. Therefore from the standpoint of the development of gravitational instability in the dust component of the cloud, conditions were far more favorable in the Jupiter zone than in the asteroid zone. For instability to develop in the asteroidal zone, particle velocities, according to (3.31), would have to be more than one order lower than in the Jovian zone;

similarly, the uniform thickness of the dust layer would have to be 1.5 orders smaller. Thus there may have been no gravitational instability at all in the asteroid zone, with simple growth of particles prevailing. If instability did develop nonetheless, it must have led to the formation of condensations of appreciably lower mass. According to (6.6), condensation masses are proportional to $\sigma^3 R^6$. Condensation masses in the asteroid zone must therefore have been two to three orders smaller than those in Jupiter's zone. Either way, Jupiter's embryo was much larger than its neighbors in the asteroid zone from the very beginning (Safronov, 1966b).

The growth process had a different outcome on the other side of Jupiter. In Saturn's zone the surface density was nearly the same, and initial condensation masses probably greater, near Jupiter. Jupiter's embryo grew more rapidly ($dm/dt \propto \sigma(t) R^{-5/2}$) and gradually outdistanced Saturn's embryo. But by that time the latter had become fairly large and the influx of bodies from the Jupiter zone did not present a threat to it. In Uranus' zone the surface density was lower and it grew more slowly. However, due to the greater R, condensations masses were greater there. The influx of bodies from the zones of Jupiter and Saturn did not interrupt the accumulation process, although it did entail considerable disruption. The anomalous inclination of Uranus' axis is likely to be due to impacts from these massive bodies.

Such in general terms are the features which characterized the process of planetary growth near the most massive bodies — Jupiter and Saturn — and which were responsible for the formation of the asteroid zone, the small mass of Mars, and the anomalous inclination of Uranus' axis of rotation. Qualitatively these features can be fully explained by means of the theory of planetary accumulation developed above. However, to check these arguments it would be necessary to study the accumulation process in this zone in greater detail, bringing in all available observational data. The asteroid belt is of great interest for cosmogony. To a large extent it preserves the features of the protoplanetary cluster of bodies inside which the planets were formed. The fragmentation products of the asteroids — meteorites — land on the Earth where they can be subjected to a variety of laboratory studies, making it possible to determine the physicochemical conditions under which these bodies formed and evolved. Comprehensive study of asteroids and meteorites thus constitutes one of the paramount tasks of planetary cosmogony (Fesenkov, 1965).

35. Rabe's theory of the formation of rapidly rotating asteroids

Rabe (1960) has suggested that rapidly rotating asteroids originated in the fusion of asteroid pairs revolving around their center of gravity. He assumes that the asteroids grew by gradually using up the finely-dispersed matter of the protoplanetary cloud. The substance which acted as a feeding medium for the asteroid embryo simultaneously served as a resisting medium. The continuous growth of the masses of the asteroid bodies in the nutrient medium and their retardation by this medium in encounters between single bodies, in Rabe's view, could have led to the formation of pairs. The initially broad, unstable pairs gradually became stable. In order for two asteroids of a pair to converge from an initial orbit relative to the center of gravity of semiaxis

360 r to complete fusion, it is necessary that the radius r of each asteroid increase 3.7 times (for a density of 2.0 g/cm^3). When two asteroids combine to form a single one, the result should be a body of elongated shape rotating at the limit of rotational stability with a period of about 5 hrs, which corresponds to the observed velocities of rotation of the asteroids. Rabe believes that pair formation in the present-day asteroid belt is nearly impossible, since growth of bodies has practically ceased there, but in the past, in his opinion, this process must have played an important role.

We will show that, despite the theoretical plausibility of Rabe's interpretation of the origin of asteroid rotation, the probability for the process of pair formation and evolution as he describes it is infinitesimal (Ruskol and Safronov, 1961).

In the two-body problem the relative velocity V_∞ of bodies before encounter, for which capture under the influence of a resisting (or nutrient) medium is possible, is given by the energy condition

$$\frac{mV_\infty^2}{2} \leqslant \int FV\,dt, \tag{1}$$

where F is the force of resistance of the medium, determined by the momentum which the body imparts to the medium in one second. It is equal to the mass of material $\pi r^2 \rho V$ encountered by the body per second, multiplied by the velocity V of the body. Consequently,

$$\frac{mV_\infty^2}{2} \leqslant \int \pi r^2 \rho V^3 dt \approx \pi r^2 \rho \overline{V^3} \Delta_0, \tag{2}$$

where Δ_0 is the diameter of the largest closed surface of zero velocity. For an asteroid of radius r and density $\delta = 2$ g/cm^3, Rabe obtains $\Delta_0 = 725\ r$ in the Sunwards direction and $\Delta_0 = 454\ r$ in the perpendicular direction. He takes the following parameters for the asteroid zone: $a_{min} = 2$ a.u., $a_{max} = 3.5$ a.u., thickness of 0.2 a.u., and total mass of matter in the zone equal to $5 \cdot 10^{24}$ g. This yields a density of $\rho = 10^{-15}$ g/cm^3. Therefore for asteroid capture it is necessary that

$$\frac{V_\infty^2}{\overline{V^2}} \leqslant \frac{2\pi r^2 \rho \Delta_0}{m} \approx 10^3 \rho \approx 10^{-12}. \tag{3}$$

The presence of a third body, the Sun, does not substantially facilitate the conditions of capture. Consequently the order of magnitude should be

$$V_\infty^2 < 10^{-12}\left(2\frac{Gm}{r} + V_\infty^2\right) \approx 10^{-12} \cdot 2\frac{Gm}{r}. \tag{4}$$

Such small relative body velocities are impossible. Mutual perturbations between asteroidal bodies increased relative velocities to a value $\approx \sqrt{Gm/\theta r}$, i.e., six orders more than necessary for capture according to (4). The influx of bodies from the Jupiter zone (see Chapter 12) further increased these velocities by one order. Thus the probability for pair formation during binary collisions in a resisting medium is very low.

However, pairs could have formed in ternary and other encounters. Statistical physics yields the following expression for the relative fraction of pairs in the case of dissociative equilibrium (Gurevich and Levin, 1950) in the semimajor axis interval da:

$$\frac{dn_2}{n_1} = 4(\pi a_0)^{3/2} e^{a_0/a} \sqrt{a} n_1 da, \tag{5}$$

where n_1 and n_2 are the number of single asteroids and pairs per unit volume and $a_0 = 3Gm/2v^2$. Body masses are assumed here to be uniform.

We obtain the fraction of binary systems with semiaxes $a_0 < a < a_2$ by integrating (5), bearing in mind that $e^{a_0/a} \approx 1$:

$$\frac{n_2}{n_1} \approx 4(\pi a_0)^{3/2} a_2^{3/2} n_1. \tag{6}$$

For the broadest pairs Rabe takes $a_2 = 360\ r$. Then

$$\frac{n_2}{n_1} \approx 2 \cdot 10^5 n_1 r^3 \approx 10^5 \rho \approx 10^{-10}. \tag{7}$$

Owing to the very low mean density ρ of matter in the asteroid zone, the fraction of asteroid pairs turns out to be very small.

Let us assume that pair formation has taken place in some way. We will show that the probability for the pattern of pair evolution suggested by Rabe (gradual convergence and fusion into a single body) is infinitesimal. Before fusion can occur the pair will disintegrate in random close encounters with other bodies (or in collisions).

Indeed, the mean disintegration time of an unstable pair ($a > a_0$), according to Gurevich and Levin, is given by

$$t_1 = \frac{3V}{16\pi G m n_1 a \ln\left(1 + \frac{a^2 V^4}{4G^2 m^2}\right)} \approx \frac{3V}{32\pi G \rho a \ln \frac{a}{2\theta r}}. \tag{8}$$

Since

$$dm = 4\pi r^2 \delta dr = \pi r^2 (1 + 2\theta) \rho V dt, \tag{9}$$

we have

$$r = r_0 + (1 + 2\theta) \frac{\rho V}{4\delta} t. \tag{10}$$

The increase in the body radius in the time t_1 amounts to

$$r_1 - r_0 = \frac{3m(1 + 2\theta)}{128\pi\theta\delta r a \ln(a/2\theta r)} \approx 10^{-2} r_0^2/a, \tag{11}$$

i. e., less than one tenth of a percent for a broad pair. However, as we saw earlier, for a pair of bodies to combine into a single body, according to Rabe, it is necessary that the body radii increase 3.7 times. Obviously, this condition is practically impossible to meet. Consequently, the fraction of asteroid pairs should be determined by the condition of dissociative

equilibrium, and from (7) it is negligible. For the early phases of evolution of the asteroid belt one can assume a density ρ two orders of magnitude greater than obtained by Rabe. But even so $n_2/n_1 \approx 10^{-8}$, i. e., the fraction of asteroid pairs is infinitesimal.

In our opinion the rotation of the asteroids and their irregular shape can be attributed in a natural way to direct collisions and fragmentations experienced by the bodies in the course of their evolution.

CONCLUSIONS

The most important characteristics of the accumulation process are the relative velocities of bodies and their size distribution. Body velocities increase due to mutual gravitational perturbations and decrease due to inelastic collisions. Simultaneous analysis of both factors (see Chapter 7) reveals that relative body velocities are conveniently defined by the expression $v=\sqrt{Gm/\theta r}$, where m and r are the mass and radius of the largest body, respectively. If the bodies have identical masses and fuse during collisions, then $\theta \approx 1$. Given a power law of mass distribution of the bodies for the terrestrial zone, $\theta \approx 3-5$. In the presence of gas θ may amount to several tens. As long as the dimensions of the largest bodies in a cluster did not exceed several kilometers, relative body velocities did not exceed 1 m/sec. Collisions among bodies took place with practically no fragmentation and ended in fusion. This result enables us to draw the important conclusion that when conditions permitting gravitational instability were absent in any given zone (as was probably the case in the portion of the dust layer close to the Sun), there could have been direct growth of bodies due to fusion in collisions.

By studying the size distribution of protoplanetary bodies by the coagulation theory method (see Chapter 8), we were able to obtain an exact solution of the equation in the absence of fragmentation for the case where the coagulation coefficient is proportional to the sum of the masses of the colliding bodies. The mass distribution function for the bodies is a product of the power function m^{-q} with exponent $q={}^3/_2$ by the exponential function e^{-bm}, which cuts off the distribution in the large mass region. The main mass of matter in this distribution is concentrated in the large bodies. Fragmentations of bodies increased the amount of fine matter in the system. Qualitative study of the coagulation equation in the presence of fragmentation makes it possible to conclude that the mass distribution function can be approximated by a power function with exponent q lying between ${}^3/_2$ and 2.

The power law is not sufficient to describe the distribution of large bodies. It was obtained without allowing for features specific to their growth. Owing to gravitation the effective collision cross-sections of the largest bodies were proportional to the fourth powers of their radii. As a result they grew at a relatively faster pace than other bodies and their orbits tended to become circular. Such bodies became potential planetary "embryos." At first there were many embryos; but as their masses and correspondingly their relative velocities increased, the source zones of adjacent embryos aggregated. The smaller of the embryos grew more slowly and apparently broke up before it could land on the larger embryo.

The number of planetary embryos decreased until the distances between them had become sufficiently large to ensure that gravitational interaction

would not be able to disrupt the stability of their orbits for a long time. This determined the law of planetary distances.

As body masses grew, so did their relative velocities. Collisions between bodies of comparable mass began to be accompanied by fragmentation. But collisions with other bodies posed no threat to planetary embryos. Their growth can therefore be described quantitatively in a completely satisfactory way by assuming that all bodies colliding with them landed on them without leading to disintegration of the embryos. Evaluation of the growth rate of planets having separate source zones (Chapter 9) leads to a growth span of 10^8 years for the Earth. The Earth has long since exhausted all the primary material in its zone, and computations of the amount of meteorite material currently landing on it cannot be utilized to evaluate its age.

The growth process of the giant planets was complicated by a number of important factors including fusion of planetary source zones, ejection of bodies from the solar system by gravitational perturbations emanating from them, and hydrogen accretion by Jupiter and Saturn (see Chapter 12). Attempts to apply the expression for the rate of body growth to the giant planets lead to serious difficulties. Given values of the parameter θ as computed for terrestrial planets and an initial surface density of solid material σ_0 computed from the present-day mass of the planets, the growth time of Uranus and Neptune proves to exceed 10^{11} years. This difficulty can be resolved by taking values one order larger for θ and σ_0, which requires one to assume that a considerable amount of solid material was ejected from the solar system in the process of planetary growth. The ejection of bodies led to the formation of a comet cloud at the periphery of the solar system which is still in existence. It would seem that the total mass of ejected bodies did not exceed one third or one half of the mass of all the giant planets together; otherwise the latter would have been drawn appreciably closer to the Sun due to preferential dissipation of bodies in the direction of revolution of the planets. Such ejection corresponds to a total initial mass of the protoplanetary gas-dust cloud of 0.05–0.06 solar masses. The loss of such large amounts of gas from the solar system could not have taken place by thermal dissipation. Effective accretion of gas by Jupiter and Saturn set in after they had attained a mass of about one to two Earth masses. An appreciable fraction of the gas in their zone had already dissipated by that time.

In the Jupiter zone the basic mass of volatile substances (CH_4, NH_3) was in the solid state and the surface density of solid material σ was several times higher than in the asteroid zone. Condensations formed in the Jupiter zone were 2–3 orders more massive than in the asteroid zone. The massive embryo which developed in the Jupiter zone began to throw bodies into adjacent zones. These bodies "washed away" most of the bodies in the asteroid zone, increasing the relative velocities of the bodies that remained. When the energy of relative motion of the bodies in that zone had become substantially larger than the potential energy at their surface, collisions among bodies began to result in fragmentation rather than fusion. Thus in the asteroid zone the accumulation process came to a standstill at an intermediate stage. The growth of Mars was also slowed down by bodies ejected from Jupiter's zone. Initial condensation masses were larger in the Saturn than in the Jupiter zone. The Jovian embryo overtook Saturn's embryo, when the latter had become comparatively large and the influx of bodies did not present a threat. In the Uranus zone condensation masses were large and the surface density was lower, the uranian embryo developing more slowly.

Incoming bodies from the zone of Jupiter (and later Saturn) were unable to interrupt the accumulation process, but they succeeded in slowing down Uranus' growth somewhat by comparison with Neptune, and were responsible for the anomalous inclination of its axis of rotation.

The bodies that landed on the planets in the process of their growth imparted rotational momentum to them. The general motion of the entire system of bodies around the Sun caused the bodies to impart to the planets a regular angular momentum component – direct rotation. In addition each individual body, having a random direction of relative velocity, also imparted a certain random angular momentum component. For the same mass of incident material, the smaller the bodies, the greater their number N; consequently, the smaller the bodies, the better the averaging of the imparted angular momenta and the smaller the mean value of the random component (which is inversely proportional to the root of N). Since the random angular momentum component was responsible for the inclination of the planets' axes of rotation, the dimensions of the largest bodies landing on a planet can be evaluated from the observed axial inclination (see Chapter 11). Computations show that the masses of the largest bodies that landed on the Earth amounted to about one thousandth of an Earth mass. The anomalous rotation of Uranus is due to the fact that its random component of rotation was greater than its systematic component owing to the large size of the bodies landing on it. The masses of the largest bodies that landed on Uranus amounted to nearly one Earth mass.

Analyzed by themselves, the equations of angular momentum and energy conservation on transition from a cluster of bodies and particles to a planet cannot furnish an explanation for the direct rotation of the planets. Direct rotation is not a result of large thermal losses in the process of planetary formation, as assumed by Shmidt. Rotation is determined by concrete conditions of collision among merging bodies, i. e., by the fundamental laws governing their motion, and it can be determined by statistical analysis of a limited three-body problem. Attempts to solve the problem numerically are encouraging.

Thus the theory of planetary accumulation from solid material furnishes us with a natural explanation, based on a unified point of view, of the principal laws of the solar system and of such characteristic features as the presence of the asteroid belt and the anomalous inclination of Uranus' axis. However, the absence of a growth theory for the giant planets which would account for all principal features of the accumulation process in their zone prevents us for the time being from giving definite answers to a number of questions, such as: How much substance was ejected from the solar system? How long did the outer planets actually take to develop? What were relative body velocities in this zone? How large were the largest bodies in the asteroid zone? Further progress in accumulation theory (especially the construction of a quantitative growth theory for the giant planets) and more extensive utilization of various kinds of observational data to check theoretical results are among the tasks now facing planetary cosmogony.

Part III

PRIMARY TEMPERATURE OF THE EARTH

Chapter 14

INTERNAL HEAT SOURCES OF THE GROWING EARTH AND IMPACTS OF SMALL BODIES AND PARTICLES

36. Warming of the Earth due to generation of heat by radioactivity and compression

By the primary temperature of the Earth we mean its temperature at the end of the formation process, which lasted over a period $\tau_p \approx 10^8$ years (see Section 27). To evaluate this quantity it is necessary to consider the three main sources of heat of the growing Earth: 1) impacts of falling bodies; 2) generation of heat by radioactivity; 3) contraction of matter due to the pressure of the layers being added at the top. The warming due to compression of matter was investigated by Lyubimova (1955). It was found that the temperature rise in compression is proportional to the temperature T of the compressed matter and is given in terms of the Grüneisen coefficient γ:

$$dT = \gamma T \frac{d\rho}{\rho}, \tag{1}$$

where $\gamma = \frac{\alpha K_s}{C_p \rho}$, α is the volumetric coefficient of thermal expansion, and K_s is the adiabatic bulk modulus.

For the Earth's mantle γ was approximated by the expression $\gamma \approx 23.5\ \rho^{-2}$. It was assumed that the core consists of metallized silicates. The distribution of density inside the Earth just before the phase transition (for $M = 0.8\,Q$) was approximated by the Roche formula. Assuming that compression of the phase transition itself (discontinuity in the core) was not accompanied by warming, Lyubimova obtained the ratio $f = T/T_0$ for different depths. Compression of matter causes the initial temperature to increase by roughly 2.3 times at the Earth's center and by 1.8 times at the boundary of the core. Warming of the Earth by radioactive heat in the course of its growth was disregarded.

In the presence of sources of radioactive heating, an additional term εdt must be inserted in expression (1):

$$dT = \gamma T \frac{d\rho}{\rho} + \varepsilon dt, \tag{2}$$

where the quantity ε (radioactive warming per unit time) can be considered constant. The equation must satisfy the initial condition

$$T(r,\ t(r)) = T_s(r) \qquad \text{for} \qquad t = t(r), \tag{3}$$

which means that at the instant $t(r)$ when the radius of the growing Earth equals r, the temperature of its surface will be $T_s(r)$. The contribution due to radioactive heat is maximum in the central region of the Earth, which was the first to form. The Earth's compression is also maximum in the center. But the initial temperature T_s is minimum in the central part. A calculation of $T_s = T_{s0}$ for an Earth bombarded by small bodies and particles, as well as an approximate calculation of T with the aid of f from the formula

$$T_1(r) \approx [T_{s0} + \varepsilon(\tau_p - t(r))]\frac{1+f}{2} \tag{4}$$

for $\varepsilon = 300°$ in 10^8 years was carried out by the author in 1958. The curve for $T_1(m^{1/3})$ is given in Figure 9.

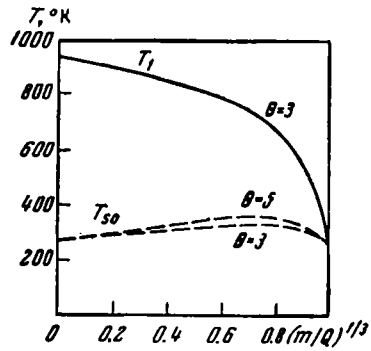

FIGURE 9. Primary temperature of Earth resulting from impacts of small bodies and particles in the accumulation process (dotted lines) and from warming due to compression of Earth material and generation of radioactive heat. Solid line shows the Earth's temperature 100 million years after it began to grow, allowing for all three heat sources for $\varepsilon = 300°$ over 10^8 years.

The Grüneisen coefficient has recently been evaluated from more recent data by Lyubimova (1968).

For the mantle

$$\gamma = \gamma_1 = \frac{a_1}{\rho} - b_1, \qquad a_1 = 6.72, \qquad b_1 = 0.13, \tag{5}$$

for the metallized silicate core

$$\gamma = \gamma_2 = b_2 - a_2\rho, \qquad a_2 = 0.146, \qquad b_2 = 2.18. \tag{6}$$

Correspondingly equation (2) for the mantle becomes

$$\frac{dT}{dt} = \left(\frac{a_1}{\rho^2} - \frac{b_1}{\rho}\right)\frac{d\rho}{dt}T + \varepsilon. \tag{7}$$

For the initial condition (3) its solution is

$$T(t_s, t) = \left(\frac{\rho_0}{\rho(t_s, t)}\right)^{b_1} e^{a_1\left(\frac{1}{\rho_0} - \frac{1}{\rho(t_s, t)}\right)} T_s + e^{-\frac{a_1}{\rho(t_s, t)}} \int_{t_s}^{t} \varepsilon e^{\frac{a_1}{\rho(t_s, t)}} \left(\frac{\rho(t_s, t')}{\rho(t_s, t)}\right)^{b_1} dt'. \quad (8)$$

Here $T(t_s, t)$ designates the temperature at the instant t of a spherical layer formed at the instant t_s. Similarly $T(m_s, m)$ will denote the temperature at the surface of a sphere containing the mass m_s for a planetary mass equal to m.

The nature of the Earth's core has yet to be established (Magnitskii, 1965). In experiments on compression by impact the corresponding transition is not detected before pressures exceeding 4 million atmospheres are reached. But from a cosmogonic point of view an iron core would pose greater difficulties than a metallized silicate one (Levin, 1962). Below we will consider a terrestrial model with a silicate core. If the core were of iron, it would have to be formed distinctly later than the Earth itself. The initial temperature would then have to be evaluated for a coreless Earth, with $\gamma \approx \gamma_1$ for the entire Earth.

The warming of the core material is given by expression (2) with the Grüneisen coefficient γ_1 before the phase transition and by the same expression with the coefficient γ_2 afterward. According to Ramsey (1949), contraction during the phase transition takes place with practically no warming (the energy is transformed into the work of deformation). This problem also awaits definitive solution, and Ramsey's assumption is best seen as a variant yielding a minimum value of the temperature. Solving equation (2) separetely for different γ and assuming that the phase transition by itself is not accompanied by warming, we obtain the following expression for the temperature inside the core:

$$T(t_s, t) = \left(\frac{\rho_0}{\rho_-}\right)^{b_1} \left(\frac{\rho(t_s, t)}{\rho_+}\right)^{b_2} e^{a_1\left(\frac{1}{\rho_0} - \frac{1}{\rho_-}\right) - a_2(\rho(t_s, t) - \rho_+)} T_s +$$

$$+ e^{-a_2(\rho(t_s, t) - \rho_+)} \int_{t_s}^{t^*} e^{a_1\left(\frac{1}{\rho(t_s, t')} - \frac{1}{\rho_-}\right)} \left(\frac{\rho(t_s, t')}{\rho_-}\right)^{b_1} \left(\frac{\rho(t_s, t)}{\rho_+}\right)^{b_2} \varepsilon dt' +$$

$$+ e^{-a_2\rho(t_s, t)} \int_{t^*}^{t} e^{a_2\rho(t_s, t')} \left(\frac{\rho(t_s, t)}{\rho(t_s, t')}\right)^{b_2} \varepsilon dt'. \quad (9)$$

Here ρ_- and ρ_+ are the density just before and after the phase transition and t^* is the time corresponding to the phase transition at the point under consideration. For all $m_s \leqslant 0.08\ Q$ the phase transition occurs when $m \approx 0.8\ Q$, while for the remaining values of m_s inside the core it occurs later.

Alternatively, one could assume that the phase transition generates the same amount of heat as given by relation (2), with γ assuming intermediate values between γ_1 and γ_2. If as in (6) we take a linear dependence of γ on ρ during the transition, we obtain

$$\gamma = 1.58 - 0.086\rho. \quad (10)$$

An expression similar to (9) is then obtained for the temperature inside the core, except that the first two terms on the right are 1.8 times larger. Indeed, if we assume that the effective γ responsible for warming in phase

transition amounts to only a fraction $\zeta < 1$ of the value (10), then the correction factor in the first two terms of (9) will accordingly be 1.8^{ζ}.

In expressions (8) and (9) it is convenient to take the mass m of the growing planet as independent variable instead of the time t. These quantities are related by (9.14):

$$\frac{dm}{dt} = \frac{4\pi(1+2\theta)}{P}\sigma_0 r^2\left(1-\frac{m}{Q}\right) = \sqrt[6]{36\pi}\,\frac{(1+2\theta)\sigma_0}{P}\,\frac{m^{2/3}\left(1-\frac{m}{Q}\right)}{\bar{\rho}^{2/3}}, \tag{11}$$

where $\bar{\rho} = \bar{\rho}(m)$ is the mean density of a planet of mass m. Relation (8) for the temperature inside the mantle can then be written as

$$\begin{aligned} T(m_s, m) &= e^{a_1\left(\frac{1}{\rho_0} - \frac{1}{\rho(m_s, m)}\right)}\left(\frac{\rho_0}{\rho(m_s, m)}\right)^{b_1} T_s + \\ &+ \varepsilon\int_{m_s}^{m}\left[\frac{\rho(m_s, m')}{\rho(m_s, m)}\right]^{b_1} e^{a_1\left[\frac{1}{\rho(m_s, m')} - \frac{1}{\rho(m_s, m)}\right]}\frac{dm'}{(dm/dt)'} = \\ &= A_1(m_s, m)\,T_s + B_1(m_s, m)\,\varepsilon\,(t - t_s). \end{aligned} \tag{12}$$

Expression (9) for the temperature of the core is modified in a similar way. Inserting the factor 1.8^{ζ} for possible generation of heat in the phase transition, we obtain the following expression for the core:

$$\begin{aligned} T(m_s, m) &= 1.8^{\zeta}\left(\frac{\rho_0}{\rho_-}\right)^{b_1}\left(\frac{\rho(m_s, m)}{\rho_+}\right)^{b_1} e^{a_1\left(\frac{1}{\rho_0} - \frac{1}{\rho_-}\right) - a_2[\rho(m_s, m) - \rho_+]}\,T_s + \\ &+ 1{,}8^{\zeta}\varepsilon\left(\frac{\rho(m_s, m)}{\rho_+}\right)^{b_1} e^{-a_2[\rho(m_s, m) - \rho_+]}\int_{m_s}^{m^*} e^{a_1\left[\frac{1}{\rho(m_s, m')} - \frac{1}{\rho_-}\right]} \times \\ &\times\left[\frac{\rho(m_s, m')}{\rho_-}\right]^{b_1}\frac{dm'}{(dm/dt)'} + \varepsilon e^{-a_2\rho(m_s, m)}\int_{m^*}^{m} e^{a_2\rho(m_s, m')} \times \\ &\times\left[\frac{\rho(m_s, m)}{\rho(m_s, m')}\right]^{b_1}\frac{dm'}{(dm/dt)'} = A_2(m_s, m)\,T_s + B_2(m_s, m)\,\varepsilon\,(t - t_s), \end{aligned} \tag{13}$$

where m^* is the body mass for which phase transition occurs at the point m_s (i.e., at the surface of a sphere containing a mass m_s).

Quantity $T(m_s, m)$ has been calculated by the author from formulas (12) and (13) for values of the density $\rho(m_s, m)$ at different points m_s inside a body of mass m and mean density $\overline{\rho(m)}$, taken from data given by Kozlovskaya (1967). Kozlovskaya computed a series of body models of different mass on a BESM-2 high-speed computer, for an equation of state corresponding to undifferentiated terrestrial material. It was assumed that $\rho_0 = 3.47$, $\rho_- = 5.41$ and $\rho_+ = 10.16$, which corresponds closely to model No. 7 for the Earth according to Pan'kov and Zharkov (1967). In this series of models the mean density of the bodies depends on their mass as follows:

m	0	0.1	0.3	0.5	0.7	0.838	0.838	0.9	1.0
$\bar{\rho}$	3.47	3.70	3.99	4.16	4.31	4.39	4.71	5.08	5.54

Numerical integration of (11) with these values of $\bar{\rho}(m)$ yields the time dependence of the growing Earth's mass depicted in Figure 10. The time required for the Earth to grow to 97% of its present mass for $\theta = 3$ is

86 million years, which confirms the value 88 million years estimated for the same θ assuming a constant density intermediate between the initial and final density ($\bar{\rho} = 4.5$). For $\theta = 5$ the duration of the growth process decreases to 55 million years.

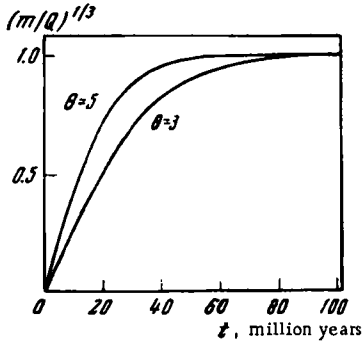

FIGURE 10. Rates of growth of the Earth for two values of the parameter θ characterizing relative velocities of bodies.

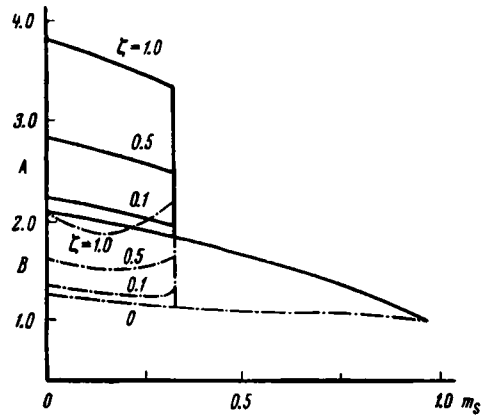

FIGURE 11. Dependence on m_s of coefficients A (solid line) and B (dash-dot line) characterizing terrestrial warming due to contraction and to the decay of radioactive elements.

Figure 11 illustrates the dependence on m_s of the coefficients A and B, defined according to (12) and (13) (subscripts have been dropped for brevity) and computed for $m = 0.97$. Table 15 lists the values of the second term $Be\,(t - t_s)$ for $\varepsilon = 200°$, $t = 10^8$ years, and $\theta = 3$ and 5.

TABLE 15

ζ	m_s								
	0	0.15	0.29	0.35	0.50	0.65	0.80	0.90	0.97
					$\theta = 3$				
0	224	158	139	133	115	96	70	48	0
0.1	234	165	147						
0.5	282	195	185						
1.0	357	245	248						
					$\theta = 5$				
0	143	101	89	85	73	61	45	27	0
0.1	149	105	94						
0.5	179	125	118						
1.0	227	156	158						

It is evident from Table 15 that radioactive heating played a relatively unimportant role during the period of terrestrial formation: the inner portion

of the mantle was warmed by roughly 100°, the core by 150 — 200°. Of greater significance is the warming due to the heating T_s of the surface of the developing Earth by impacts from falling bodies and to the A-fold rise in this temperature resulting from compression of material in the Earth's interior. For $m_s \sim 0.5$ one can assume that the minimum value of T_s is given by $T_s \approx 350°$. Then $AT_s \approx 600°$, i. e., it exceeds the heating $B\varepsilon(t-t_s)$ due to radioactivity by a factor of five or more. Therefore if one wishes to arrive at a more accurate value for the initial temperature it is most important to correct the value of T_s, the temperature at the surface of the developing Earth.

The energy spent in warming the Earth due to contraction of its material is small compared with the total energy of compression. Energy is expended chiefly in the deformation of material. Lyubimova (1962) has evaluated the energy expended in the elastic deformation of a homogeneous sphere of terrestrial mass under the influence of its gravitational field. According to the model used in the calculations, a nongravitating, undeformed Earth is formed initially; its gravitational field is then included, with the corresponding deformation. Depending on values used for the parameters, the estimated energy of deformation varies between the limits $(5-9) \cdot 10^{38}$ erg, i. e., it amounts to a considerable fraction of the potential energy of the Earth as a sphere (Lyubimova, 1968). If an appreciable fraction of this energy had been contained in shearing stresses and had been liberated in the process of the Earth's evolution upon relaxation of these stresses, it could have been an important source of internal energy. In reality the Earth's development was gradual as was the intensification of its gravitational field, and the deformations increased gradually. Allowance for this should yield a smaller energy of deformation. Another element of inaccuracy in the calculations is the insufficient reliability of the numerical values used for the parameters characterizing the elastic properties of the Earth's material, making it impossible to draw definite conclusions.

In this connection it is of interest to calculate directly the energy of compression of the Earth during its growth, which represents that fraction of the Earth's energy of deformation which cannot be liberated in the relaxation of elastic stresses and cannot convert into heat. To evaluate the energy of compression it is sufficient to know the equation of state $P(\rho)$ and the density distribution inside the Earth. Quantity $P(\rho)$ can be approximated in the form

$$P = a\rho^n - b. \tag{14}$$

Then the energy of compression per unit mass is given by

$$W(\rho) = \int P \frac{d\rho}{\rho^2} = \int_{\rho_0}^{\rho} \frac{a\rho^n - b}{\rho^2} = \frac{a}{n-1}(\rho^{n-1} - \rho_0^{n-1}) - b\left(\frac{1}{\rho_0} - \frac{1}{\rho}\right) \tag{15}$$

and the total energy of compression by

$$W = 4\pi \int W(\rho)\rho r^2 dr. \tag{16}$$

The approximation of the curve of $P(\rho)$ obtained by Kozlovskaya (1966) for $\rho_0 = 3.47$ gives the following values for the mantle: $n = {}^{13}/_{3}$, $a = 1.05 \cdot 10^9$, and $b = 2.4 \cdot 10^{11}$. Integrating (16), we then find that the energy of compression

of the mantle material is $W_{man} = 0.6 \cdot 10^{38}$ erg. The material of the silicate core obeys the same equation of state as the mantle material up to the phase transition, after which one can take an equation of the form (14) with $n' = 3$, $a' = 2.2 \cdot 10^9$, and $b' = 1.0 \cdot 10^{12}$. The total compression energy of the core material is found to be $3.6 \cdot 10^{38}$ erg. Of this, $0.9 \cdot 10^{38}$ erg goes for compression before phase transition, $2.3 \cdot 10^{38}$ erg for compression during and $0.4 \cdot 10^{38}$ for compression after the phase transition. Thus the entire compression energy of the Earth's material amounts to

$$W = W_{man} + W_{core} = 0.6 \cdot 10^{38} + 3.6 \cdot 10^{38} = 4.2 \cdot 10^{38} \text{ erg}. \qquad (17)$$

The total energy of deformation should not be less than the above energy of compression. It is interesting to note that over half of the compression energy is expended in phase transition in the core and only $^1/_7$ in contraction of the mantle material. Thus we see how important it is to know the change in thermal energy which occurs in the phase transition. The conversion into heat of a mere 10% of the energy expended in the transition would have led to warming of the core material by 1000°.

The foregoing estimates were based on the assumption of a silicate core that has gone over into a metallized state. The problem of terrestrial thermal processes must be stated differently if one assumes an iron core. According to an estimate by Lyustikh (1948), about $1.5 \cdot 10^{38}$ erg should convert into heat when iron overflows into the core from the mantle, corresponding to warming of the material of the whole Earth by 2400°.* Assuming that this overflow was possible (which would require the existence of large inclusions of metallic iron), it must have occurred after the Earth had formed, when it had warmed up sufficiently. This applies to the differentiation of all substances in the Earth, the total energy of which may have been considerable (see, for instance, Krat (1960)) and should be taken into account when studying the thermal history of the Earth. Unfortunately there are no definite data on the scale of differentiation.

37. Warming of the Earth by impacts of small bodies and particles

Formulas (13.12) and (13.13) for the initial temperature of the Earth contain the temperature T_e of the surface of the growing planet. It is determined by the energy of impacts from bodies falling on the Earth during its formation, and moreover depends on the dimensions of these bodies. The simplest way of estimating T_e is by assuming that the Earth was formed from small bodies and particles. We will denote the surface temperature for this case by T_{e0}. Bodies can be regarded as small in the problem under consideration if the energy liberated when they fall on the Earth is liberated

* Owing to the low rate of settling of the heavier inclusions, their kinetic energy is negligible (even for a viscosity of 10^{17} poise and inclusion radius of 100 km, the Stokes velocity will not exceed 1 cm/sec). The potential energy liberated by the inclusions as they settle down should therefore convert into heat throughout the Earth's sphere, without leading to preferential warming of the core compared with the mantle. An iron core impoverished in radioactive elements could not subsequently become warmer than the lower mantle. This makes it difficult to explain the Earth's magnetic field, which is usually related to convective motions in the liquid outer part of the core.

in the immediate vicinity of the surface and is almost entirely irradiated into space. A layer of thickness h warmed on impact will cool within an interval of the order of h^2/k, where k is the coefficient of thermal conductivity. The time required for the laying down (due to the Earth's growth) of a new layer of material of thickness h is given by $h/\dot{r}$. If the latter is less than the former, the greater part of the layer's heat will remain inside the Earth. Bodies can therefore be termed small if the thickness of layer warmed upon their settling is

$$h \ll k/\dot{r}. \tag{18}$$

From (9.14) it can be shown that the rate of increase in the Earth's radius $\dot{r} \sim 10^{-7}$ cm/sec. For the usual molecular thermal conductivity $k \sim 10^{-2}$, we obtain $h \ll 1$ km. Since the thickness of the layer warmed on impact is of the order of the diameter of the fallen body, it follows from (18) that all bodies with diameters less than a hundred meters can be classed among small bodies; the energy of their fall is almost entirely emitted into space. In Section 40 it will be shown that due to the considerable mixing of the material by impacts from falling bodies, the effective thermal conductivity was 2 — 3 orders greater than the molecular thermal conductivity. The dimensions of the bodies whose energy of fall was trapped inside the Earth must have been correspondingly larger as well.

Suppose the Earth's growth took place as a result of the fall of small bodies (in the above sense) and that the energy they imparted was liberated practically at the surface. Without introducing a serious distortion we can assume that the rate of increase of the Earth's radius $\dot{r}$ was constant and that its surface was flat. In this case the surface temperature T_{s0} is independent of the time and can be determined from a simple relation expressing the equality of the energy brought by the bodies and the energy emitted:

$$\left(\frac{Gm}{r}+\frac{v^2}{2}\right)\frac{dm}{dt}=4\pi r^2\sigma'(T_{s0}^4-T_0^4)+c(T_{s0}-T_p)\frac{dm}{dt}. \tag{19}$$

Here m and r are the mass and radius of the growing Earth, T_p the temperature of the falling bodies and particles, v their mean velocity with reference to the Earth before encounter, T_0 the black-body temperature near the Earth, σ' the Stefan-Boltzmann constant, and c the heat capacity. Temperatures are reckoned from absolute zero. The left-hand side of (19) represents the energy imparted to the Earth by falling material per unit time. The first two terms on the right represent the energy lost by the Earth due to emission (emission minus absorption). The last two terms on the right are the energy expended in warming terrestrial material. They are nearly two orders smaller than the others and can be totally disregarded, since the main term in (19) contains T_{s0} in the fourth power. Substituting for dm/dt from (9.14) and inserting $v^2=Gm/\theta r$, we obtain (Safronov, 1959)

$$T_{s0}^4=T_0^4+\frac{(1+2\theta)^2\,\sigma_0 Gm\,(1-m/Q)}{2\theta\sigma' Pr}. \tag{20}$$

The values of T_{s0} obtained from this expression for different m are given in Figure 9. They are maximum for layers now situated at a depth of 2 — 2.5 thousand km, and for $\theta = 3 - 5$ they amount to about 350 — 400°K.

The low gradient of T_{s0} over r validates the presumption of stationariness and means that the surface temperature T_{s0} of the growing Earth was simultaneously the temperature of its material before appreciable amounts of heat had been liberated inside it due to radioactivity and contraction.

Chapter 15

WARMING OF THE EARTH BY IMPACTS OF LARGE BODIES

38. Thermal balance of the upper layers of the growing Earth

When bodies settle on the Earth, most of the energy of impact is liberated inside a layer having a thickness of the order of the diameter of the fallen body. The surface temperature fluctuates sharply in the process, and its mean value is slightly less than the value obtained earlier for T_{e0}, since $\bar{T}_{sur} < \sqrt[4]{\overline{T^4_{sur}}} = T_{e0}$. In order for heat to escape from the layer warmed by the impacts into the open, there must be a negative temperature gradient along r. Consequently, the thicker the layer, i.e., the larger the falling bodies, the higher the temperature of the material under the layer. On the other hand, the larger the fallen body, the larger the crater it produces and the greater the depth of mixing during impact. Heat transfer by mixing of material during the fall of large bodies is far more efficient than heat transfer by ordinary thermal conduction (molecular, radiant, etc.). As there is no theory yet which would permit us to allow for the specific character of mixing by impact, it is natural to seek to use the methods of the theory of heat conduction. To do this we must determine the appropriate value of the analog of the coefficient of thermal conductivity K associated with mixing, and the depth distribution of the heat sources $\mathcal{E}$.

On the whole the problem of the Earth's warming by impacts of falling bodies is fairly complicated, as it involves setting up and solving the equation of thermal conduction (more precisely, of heat transfer) for a spherical volume having a moving boundary with an adjacent region of heating and intensive mixing. To determine the quantities K and $\mathcal{E}$ entering into this equation, one must in turn know; a) the shape and size of the craters formed during impacts; b) the fraction of energy expended in warming the material under the crater and its depth distribution; c) the mass distribution of bodies from which the Earth was formed.

The data available on these questions are unfortunately highly unreliable. In particular, at present there exists no sufficiently complete theory of cratering, and almost nothing is known of the results of the fall of very large bodies, where the force of the Earth's gravity disrupts geometric similitude substantially, leading to qualitatively new phenomena. We will therefore have to confine ourselves to the simplest schemes if we wish, first, to evaluate the role of the main factors in the first approximation and, second, to determine whether the falling of large bodies on the Earth could have led to an appreciably higher initial temperature than obtained in Chapter 14.

Owing to the random character of impacts due to the infrequently falling large bodies, K and $\mathcal{E}$ varied sharply in space and time, leading to uneven warming of the Earth. The question of initial thermal inhomogeneities is discussed in Chapter 16. In the present chapter we consider only the mean values of K and $\mathcal{E}$, calculating therefore a mean smoothed initial temperature of the Earth. Our first step will be to derive an initial equation and obtain its solution; next we will seek to calculate K and $\mathcal{E}$.

Below it will be shown that K varies appreciably with depth. Although the general equation of thermal conduction for K dependent on x remains linear as before, in practice it is complicated to obtain a solution satisfying the required initial and boundary conditions, especially if the boundary is moving. In calculating the warming of the upper layers of the growing Earth by impacts from falling bodies, it is therefore expedient to confine oneself to the simpler case of the stationary state of a plane half-space whose boundary is moving with a constant velocity $dr/dt = \dot{r}$.

In a reference system bound to the material, the equation of thermal conduction has the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \mathcal{E} = K\frac{\partial^2 T}{\partial x^2} + \frac{\partial K}{\partial x}\frac{\partial T}{\partial x} + \mathcal{E}, \tag{1}$$

where x is reckoned from some initial position of the surface. Quantities K and $\mathcal{E}$ are functions of the distance z from the actual position of the surface:

$$K = K(z), \qquad \mathcal{E} = \mathcal{E}(z), \qquad z = x + \dot{r}t. \tag{2}$$

It is therefore natural to pass from the independent variable x in equation (1) to the independent variable z. Then

$$T(x, t) \equiv T^*(z, t); \quad \frac{\partial T}{\partial t} = \frac{\partial T^*}{\partial t} + \dot{r}\frac{\partial T^*}{\partial z}; \quad \frac{\partial T}{\partial x} = \frac{\partial T^*}{\partial z}. \tag{3}$$

Obviously, the steady solution should also be sought in a moving coordinate system: $\partial T^*/\partial t = 0$. For brevity we drop the asterisk ($T^*(z, t) \equiv T(z)$). Then the transformed steady equation of thermal conduction takes the form

$$K\frac{\partial^2 T}{\partial z^2} + \left(\frac{dK}{dz} - \dot{r}\right)\frac{dT}{dz} + \mathcal{E} = 0. \tag{4}$$

The solution of this equation should satisfy the boundary conditions:

$$\begin{aligned} T(0) &= T_{s0} && \text{for } z = 0 \\ dT(\infty)/dz &= 0 && \text{for } z = \infty \end{aligned} \tag{5}$$

Let us set

$$K\frac{dT}{dz} = u. \tag{6}$$

Then

$$u' - \frac{\dot{r}}{K}u + \mathcal{E} = 0 \tag{7}$$

and

$$u = e^{y}\left(C - \int e^{-y}\mathcal{E}dz\right) = e^{y}\int_{y}^{\infty} e^{-y}\mathcal{E}dz, \tag{8}$$

where

$$y = \int_{0}^{z} \frac{r}{K}\,dz. \tag{9}$$

Owing to the second boundary condition, $u(\infty) = 0$ and therefore $C = 0$. Let us now calculate T:

$$T = \int \frac{u}{K}\,dz = \int K^{-1}e^{y}dz \int_{r}^{\infty} e^{-y}\mathcal{E}dz + C.$$

From the first boundary condition,

$$T = \int_{0}^{z} K^{-1}e^{y}dz \int_{z}^{\infty} e^{-y}\mathcal{E}dz + T_{\infty}. \tag{10}$$

Expressing z in terms of y and changing the order of integration, we reduce the double integral to single integrals:

$$\int_{0}^{z} K^{-1}e^{y}dz \int_{z}^{\infty} e^{-y'}\mathcal{E}(z')\,dz' = \frac{1}{r^2}\int_{0}^{y} e^{y}dy \int_{y}^{\infty} e^{-y'}\mathcal{E}(y')\,K(y')\,dy' =$$

$$= \frac{1}{r^2}\left[\int_{0}^{y} e^{-y'}\mathcal{E}(y')\,K(y')\,dy' \int_{0}^{y'} e^{y}dy + \int_{y}^{\infty} e^{-y'}\mathcal{E}(y')\,K(y')\,dy' \int_{0}^{y} e^{y}dy\right] =$$

$$= \frac{1}{r^2}\left[\int_{0}^{y} (1 - e^{-y'})\,\mathcal{E}(y')\,K(y')\,dy' + (e^{y} - 1)\int_{y}^{\infty} e^{-y'}\mathcal{E}(y')K(y')\,dy'\right].$$

Passing from y back to z, we obtain the expression for T:

$$T(z) = \frac{1}{r}\left[\int_{0}^{z} (1 - e^{-y})\,\mathcal{E}dz + (e^{y} - 1)\int_{z}^{\infty} e^{-y}\mathcal{E}dz\right] + T_{\infty}. \tag{11}$$

The primary Earth temperature T_{e} due to its warming by the impacts of falling bodies is clearly

$$T_{\text{e}} = T(\infty) = \frac{1}{r}\int_{0}^{\infty} (1 - e^{-y})\,\mathcal{E}dz + T_{\infty}. \tag{12}$$

39. Fundamental parameters of impact craters

In order to calculate quantities K and $\mathcal{E}$ in equation (12), it is necessary to know the size of the crater formed by the fallen body, the thickness of the layer settling around the crater, and the depth distribution of the energy liberated on impact. We will assume that the fall of a body of radius r' will

form a cylindrical crater whose "original" depth h and "original" radius R are proportional to r':

$$h = \nu_1 r', \qquad R = \nu_2 r'. \tag{13}$$

The term "original" refers to the size of the crater before its sides collapse and before part of the ejected material falls back into the crater. Below we will show that for a uniform rate of fall, ν_1 can be considered constant while ν_2 decreases slowly with increasing r'. Salisbury and Smalley (1964) assume on the basis of laboratory studies (Gault et al., 1964) that for an impact velocity of 11 km/sec the ejected mass will be 10^3 times greater than the mass m' of the fallen body. Crater depth at this speed amounts to about two diameters of the falling body (Öpik, 1958; Bjork, 1961; Andriankin and Stepanov, 1963). Consequently one can take $\nu_1 \approx 4$ and $\nu_{20} \approx 18$, where ν_{20} is the value of ν_2 for small r'.

Suppose further that the material thrown out of the crater evenly covers an area in a circle of radius R_1 with a layer of thickness h_1:

$$h_1 = \frac{R^2}{R_1^2} h; \tag{14}$$

R_1 is determined by the rate of ejection of material from the crater.

Assuming that in the propagation of shock waves the energy of motion is constant and identical in all directions ($Mv^2 = \text{const}$, where M is the mass affected by the explosion), Stanyukovich (1960) obtains the following distribution of the velocity of the ejected matter:

$$v = v_0 \left(\frac{r'}{R'}\right)^{3/2}, \tag{15}$$

where v_0 is the impact velocity of a meteorite of radius r'. The velocity v characterizes all particles on a ray of length $R' = \sqrt{R^2 + w^2}$ beginning at the center of the explosion at depth w and ending at the surface at distance R from the epicenter. Arguments in favor of this result for large impact velocities are also cited by Andriankin and Stepanov (1963). On the other hand, Lavrent'ev (1959) and Pokrovskii (1964) hold that the momentum remains constant and $v \propto R'^{-3}$. Basing himself on his solution of the problem of concentrated impact for a single simplified model, Raizer (1964) concludes that the velocity law is appreciably closer to $R'^{-3/2}$ in this case than to R'^{-3}. Dokuchaev, Rodionov and Romashov (1963) give the velocity law $v \propto R'^{-1.8}$, obtained by measuring the maximum rate of dispersion of particles as a cupola rises over the site of a deep explosion. This law is also closer to the case of energy conservation than to that of momentum conservation. For the impact velocity of interest to us (10 — 12 km/sec) the amount of material which evaporates is comparatively small and the energy of expansion of the resulting gases plays a relatively unimportant role. Thus Shoemaker (1962) found from data given by Altschuler regarding the equation of state of iron that the main meteorite mass melts only when the impact velocity reaches 9.4 km/sec. But as far as intensity of pressure is concerned, impacts having such velocities are similar to explosions intermediate in character between chemical and nuclear explosions.

At low explosion depths w the dispersion rate v obtained from the relations of Dokuchaev and others (especially Pokrovskii) is too high. In this respect expression (15) is to be preferred. Let us take

$$v = v_0\left(\frac{r'}{R'}\right)^q. \tag{15'}$$

The total energy of disperion of material for a conical crater of depth $h \approx w$ is given by

$$E_s = \int \frac{v^2}{2} dM = \frac{v_0^2}{2}\int_0^R \left(\frac{r'}{R'}\right)^{2q} \frac{2\pi}{3} \rho w R dR = \frac{v_0^2}{2} \cdot \frac{4}{3}\pi\rho r'^3 r'^{2q-3} \times \int_0^R \frac{wRdR}{2(w^2+R^2)^q} = \eta m' \frac{v_0^2}{2}. \tag{16}$$

From this we obtain the ejection efficiency coefficient η:

$$\eta = \left(\frac{r'}{w}\right)^{2q-3} \frac{1}{4(q-1)}\left[1 - \frac{1}{(1+n^2)^{q-1}}\right], \tag{17}$$

where $n = R/w$ is the ejection index. Table 16 lists numerical values of η.

TABLE 16

n	q		
	1.5	1.8	3.0
	$w = 4r'$		
2.0	0.28	0.10	0.002
4.5	0.39	0.12	0.002
	$w = 3r'$		
2.0	0.28	0.12	0.004
4.5	0.39	0.15	0.005

The rows with $n = 4.5$ and $n = 2$ apply to the fall of small and large bodies, respectively. According to Dokuchaev and others, in deep explosions in clay n increases from 0.11 to 0.14 as η varies from 1.7 to 2.8 (see Figure 47 in their work). Closest to these figures in Table 16 are the values of η for $q = 1.8$.

Expression (15') makes it possible to evaluate the distance R_1 of maximum dispersion if the direction of the dispersion velocity is known. Usually the initial direction of the velocity from the center of the explosion is taken to be radial (as an approximation). According to Dokuchaev and others, for deep explosions ($n \approx 1.5 - 2.5$) the initial velocities of points on the surface are directed, in the first approximation, radially from a point situated at twice the explosion depth ($2w$). However, the authors believe that their conclusions cannot be extended to explosions with relatively little hollowing (i. e., with large n). We will take as R_1 (approximately) the distance of dispersion of particles whose radii vectors are directed from the center of the explosion at an angle of 45° to the horizon. Then

$$R_1 = w + \Delta R = w + \frac{v^2}{g} = w + \frac{v_0^2}{g}\left(\frac{r'}{w\sqrt{2}}\right)^{2q} \approx \nu_1 r' + \frac{v_0^2}{g(\nu_1\sqrt{2})^{2q}}. \tag{18}$$

For a vertical impact with a velocity of 10 — 12 km/sec one can take $v_1 \approx 4$; for an impact at an angle of 45°, $v_1 \approx 3$. Then for $q = 1.8$ the maximum distance ΔR from the take-off point is found to be 24 and 68 km, respectively.

The dimensions of the crater are determined by the energy and depth of the explosion. With regard to the depth of penetration of the body w ("explosion center"), and therefore the depth h of the original crater, which is slightly larger than w, geometric similitude holds: for the same impact velocity they are proportional to the radius r' of the falling body (i. e., to the cube root of the impact energy). Therefore the "relative explosion depth" $w/C^{1/3}$, as it is usually taken in blasting (with the power $^1/_3$ on the charge C), is independent of the size of the falling bodies and for $w = 4r'$ and velocity $v_0 = 11$ km/sec it equals 0.07 m/kg$^{1/3}$, which corresponds to small, near-contact explosions. The power 1/3, however, is probably suited only to very small craters not more than a few meters deep, where the energy which must be expended in the formation of the crater is proportional to its volume $R^2 w = n^2 w^3$ (overcoming atmospheric pressure, destruction of matter). In the case of large bodies, an additional, considerable amount of energy is expended in overcoming gravity to lift the material out of the crater. This energy is proportional to $R^2 w \cdot w = n^2 w^4$, and its relative importance increases with increasing $w \propto r'$. The geometric similitude with respect to the crater dimensions is therefore destroyed. On the basis of extensive data on chemical and nuclear reactions, American specialists have concluded that the linear dimensions of the crater increase in proportion to the explosion energy in the power 1/3.4 (Shoemaker et al., 1961; Nordyke, 1962). Assuming radial dispersion and disregarding the resistence of the material, Pokrovskii (1964) found, from the condition that the ejection rate along the crater slope was such that the material was thrown out only over the crater's edge, that such craters (with the same $n = R/w$) are obtained for constant $w/C^{1/3.4}$. From experiments concerning the expansion of a gas bubble in sand contained in a vacuum, Sadovskii, Adushkin and Rodionov (1966) concluded that n must depend on $w/C^{1/4}$. For an exponent 1/3.4, $n \propto (r')^{\frac{3}{3.4}-1} = (r')^{-\frac{1}{8.5}}$. Pokrovskii's relation gives $n \propto (r')^{-\frac{1}{8}}$ for $n^2 \gg 1$. The relation of Sadovskii and others is logarithmic. It was obtained for the interval $0.6 < n < 2.6$, and its form for large n of interest to us is unknown. One can apparently take $n \propto r'^{-\alpha}$ for $r' > r_0$ and

$$\frac{R}{r'} = v_2 = v_{20}\left(\frac{r_0}{r'}\right)^{\alpha} \quad \text{for} \quad r' > r_0, \qquad (19)$$
$$v_2 = v_{20} \quad \text{for} \quad r' < r_0,$$

where $\alpha \ll 1$ and r_0 is of the order of a few meters.

Since the relative explosion depth should be determined by the ratio $w/C^{1/\mu}$, where $3 < \mu < 4$, it should increase with the size of the falling bodies: $w \propto r'$; $C \propto r'^3$ and $w/C^{\frac{1}{\mu}} \propto (r')^{1-\frac{3}{\mu}}$. As r' increases, the explosions become relatively deeper and n decreases. For large enough $r' \geqslant r_1$ a "loosening explosion," in which practically all the material thrown out falls back into the crater, may take place. If the function $n = f\left(\frac{w}{C^{1/\mu}}\right)$ extends to explosions that qualify as relatively small (with respect to the magnitude of $w/C^{1/3}$), it is easy to determine how much one needs to add to the absolute depth w in order to obtain explosions with a specific n for a small value of $w/C^{1/3}$:

$$n = f\left(\frac{w}{C^{1/\mu}}\right) = f_1\left(\frac{w^{\mu/3}}{C^{1/3}}\right) = f_1\left(\frac{w}{C^{1/3}} w^{\frac{\mu}{3}-1}\right). \tag{20}$$

At depths of 10 — 30 m loosening explosions occur when $w/C^{1/3} \geqslant 1$. For falling bodies this ratio is 15 — 20 times smaller. Therefore to obtain the same value of n, $w^{\mu/3-1}$ should, from (20), be increased as many times. For $\mu = 4$ the value of n will be the same for falling bodies as for loosening explosions with $w/C^{1/3} = 1$ and $w = 20$ m provided $w \approx 60$ and 150 km, and if $\nu_1 = 4$ and 3. Impacts similar to loosening explosions should therefore take place on the falling of bodies with radius $r_1 \geqslant 15$ and 50 km, respectively. In instances where $\mu = 3.4$ the size of bodies which produce impacts similar to loosening explosions should be many times larger. However, one can expect that at such considerable depths the role of gravitation becomes dominant, with μ approaching 4. Baldwin (1963) concludes on the basis of studies of the parameters of lunar craters that μ increases with the size of falling bodies, reaching 3.6 for craters 10 miles in diameter. But his claim that μ begins to decrease for yet larger craters is unfounded.

The condition for transition to loosening can be obtained directly from energy considerations. The ejection from a crater of the material of mass M contained inside it involves expenditure of the energy

$$E_{ej} \geqslant Mgh_e \approx Mg\frac{w}{2} \approx \frac{\nu_1}{2} Mgr', \tag{21}$$

where $h_e \approx w/2$ is the minimum height to which the center of gravity of the mass M must be raised for it to be thrown out of the crater. Since

$$E_{ej} = \eta m' v_0^2/2 \quad \text{and} \quad M/m' \approx \frac{R^2 w}{2r'^3} \approx \frac{n^2 \nu_1^3}{2}$$

(for a spheroidal crater), the size of a body forming a crater with index n should satisfy the condition

$$r' \leqslant \frac{2\eta v_0^2}{\nu_1^4 n^2 g} = \frac{4\eta}{\nu_1^4 n^2} r,$$

where r is the Earth's radius. For $n = n_1$ an ejection explosion will grade into a loosening explosion. Therefore

$$r_1 \leqslant \frac{4\eta r}{\nu_1^4 n_1^2}. \tag{22}$$

It is usual to take $n_1 \approx 1$, and it seems one can assume that $n_1 > 0.7$. Then for the present terrestrial radius and $\nu_1 = 4$, one obtains $r_1 \leqslant (100 - 200)\,\eta$ km, while for $\nu_1 = 3$ it is three times greater. If departures from geometric similitude due to the important part played by gravity cause the impacts of large bodies to resemble deep explosions in all respects (in contrast with small body impacts, which are similar to contact explosions), the ejection efficiency coefficient can be taken to be about 0.10 — 0.15 (according to Dokuchaev). Then $r_1 \leqslant 10 - 30$ km for $\nu_1 = 4$ and $r_1 \leqslant 30 - 90$ km for $\nu_1 = 3$. These values agree entirely with results obtained earlier from other considerations.

Thus one may conclude that for falling bodies with radius $r' > r_1$, where r_1 equals a few tens of kilometers, the impacts will be similar to loosening explosions. Impacts with loosening are of interest inasmuch as they warm the Earth most effectively (nearly all the heat of the fallen body remains buried inside the filled-in crater).

40. Heat transfer in mixing by impact and depth distribution of the impact energy

The fundamental relation defining the coefficient of thermal conductivity K is the well-known expression which relates it to the flux of heat:

$$H(x) = K(x)\frac{\partial T}{\partial x}. \tag{23}$$

Here for the sake of convenience $H(x)$ denotes the "temperature flux," which differs from the heat flux by a factor $1/c\rho$ and is directed upward toward the surface (i.e., toward decreasing x). Thus the evaluation of K can be reduced to the calculation of the heat transported across a unit surface at depth x during impacts from falling bodies.

Consider first the effect of a single impact leading to the formation of a crater of depth h. In (23) x is reckoned in a fixed coordinate system. Since K and T depend on the distance z to the surface, which shifts with time due to the Earth's growth, it is convenient to choose x to be identical with z at the instant of impact. Henceforth we will write z in place of x but evaluate the flux across a surface fixed in the x system. This precaution is unimportant in practice, as the mean rate of displacement of the Earth's surface (from which z is reckoned) is 2 — 3 orders less than the rate of filling of the crater (the ejected mass being 2 — 3 orders greater than the mass of the incident bodies).

In the formation of a crater of depth h, the amount of heat (difided by $c\rho$) transported upward together with the ejected material across an area of 1 cm^2 at depth z is given (assuming a linear march of $T(z)$) by

$$(h-z)\,T\left(\frac{h+z}{2}\right) \approx (h-z)\left[T(z) + \frac{h-z}{2}\frac{\partial T}{\partial z}\right]. \tag{24}$$

The amount of heat carried inward across the same area during the filling-up of the crater (without the energy $\mathcal{E}$ imparted by the impact) is given by

$$(h-z)\,\bar{T}(h'/2), \tag{25}$$

where $\bar{T}(h'/2)$ is the mean temperature of the material filling the crater, i.e., averaged over all other craters (different depths h') contributing material to the filling of the given crater. The resulting heat transported across a small area at depth z due to cratering at a depth $h > z$ is given by

$$(h-z)\left[T(z) - \bar{T}(h'/2) + \frac{h-z}{2}\frac{\partial T}{\partial z}\right]. \tag{26}$$

To evaluate $\bar{T}(h'/2)$ it is necessary to know the rate of formation of craters of various sizes and the area covered by ejected material.

Let $n(r')$ be the distribution function of bodies falling on the Earth, i.e., the number of bodies per unit interval of radius r' falling over the entire Earth per unit time. For the power distribution law $n(r') = C'r'^{-p}$ the fraction of Earth surface covered per second by craters produced by bodies with radii between r' and $r' + dr'$ is given by

$$s\,dr' = \frac{v_2^2}{4r^2} r'^2 n(r')\,dr' = \frac{v_2^2 C'}{4r^2} r'^{2-p} dr', \tag{27}$$

where r is the Earth's radius. Here s represents the mean frequency of ejections of the given scale at any point on the Earth's surface. The constant C' can be expressed in terms of the rate of growth $\dot{r}$ of the Earth's radius:

$$\frac{dm}{dt} = \int_0^{r_M} n(r') \frac{4}{3} \pi \delta' r'^3 dr' = \frac{4}{3} \frac{\pi \delta' C'}{4-p} r_M^{4-p} = 4\pi r^2 \delta \dot{r}$$

and

$$C' = \frac{3(4-p) r^2 \delta \dot{r}}{\delta' r_M^{4-p}}, \tag{28}$$

where δ and δ' are the mean density of the Earth and of the bodies incident on it, respectively.

The ejection due to a body r' will blanket an area $\pi R_1^2 - \pi R^2$ outside the crater R with a layer of thickness h_1 (see (14)). The rate of blanketing by such ejections outside the craters produced by them is given by

$$s_1 dr' = \frac{R_1^2 - R^2}{4r^2} n(r')\,dr'. \tag{29}$$

The mean value $\bar{T}(h'/2)$ for a filled-in layer of thickness $h - z$ can be written as

$$\bar{T}(h'/2) \approx \frac{\int_0^{\xi(h-z)} T\left(\frac{v_1 r'}{2}\right) h_1 s_1 dr'}{\int_0^{\xi(h-z)} h_1 s_1 dr'}, \tag{30}$$

where the upper limit $\xi(h-z)$ is determined from the condition that within the time of filling of the layer $h - z$, no body larger than $\xi(h-z)$ actually manages to fall. One could take the mathematical expectation for such a fall to be, for instance, 0.5. Then ξ can be determined from the relations

$$\int_0^{\xi(h-z)} h_1 s_1 dr' \Delta t = h - z; \quad \int_{\xi(h-z)}^{r_1} s_1 dr' \Delta t = 0.5. \tag{31}$$

In "loosening" impacts, which occur for $r' > r_1$, the calculation of $K(z)$ is slightly different from its calculation for ordinary cratering impacts. The transition from cratering to loosening is gradual. But to simplify the

calculations we will assume that as long as $r' < r_1$, a normal crater is formed and does not become filled up by ejected material; for $r' > r_1$ all ejected material falls back into the crater. Correspondingly, we will have $K(z) = K_1(z) + K_2(z)$, where $K_1(z)$ is due to bodies with $r' < r_1$ and $K_2(z)$ to bodies with $r' > r_1$. Relations (31) were written for the former kind of impacts; for the second integral the upper limit should be r_1. The expression for $\xi(h-z)$ proves very cumbersome. For $p < 4 - 2\alpha$ it can be approximated satisfactorily as follows:

$$\xi(h-z) \approx \left[\frac{(4-p-2\alpha)\,2\Delta R^2}{(p-1)\,\nu_1 \nu_{20}^2 r_0^{2\alpha}}(h-z)\right]^{\frac{1}{3-2\alpha}}. \tag{32}$$

This approximation is justified by the fact that, as will be shown below, the term $\bar{T}(h/2)$ in (26), which depends on $\xi(h-z)$ (according to (30)), is small compared with the other terms. In the linear approximation

$$T(h/2) = T\left(\frac{\nu_1 r'}{2}\right) = T(z) + \left(\frac{\nu_1 r'}{2} - z\right)\frac{\partial T}{\partial z}$$

and

$$\bar{T}(h/2) = T(z) + \frac{\partial T}{\partial z}\left[\frac{\nu_1}{2}\,\frac{\int\limits_0^{\xi(h-z)} h_1 s_1 r' dr'}{\int\limits_0^{\xi(h-z)} h_1 s_1 dr'} - z\right]. \tag{32'}$$

Using (26), (30') and (32), we obtain the contribution of bodies between r' and $r' + dr'$ to $K_1(z)$:

$$dK_1(z) \approx (h-z)\left[\frac{h+z}{2} - \frac{\nu_1}{2}\,\frac{\int\limits_0^{\xi(h-z)} h_1 s_1 r' dr'}{\int\limits_0^{\xi(h-z)} h_1 s_1 dr'}\right] s dr'. \tag{33}$$

To find $K_1(z)$ one must integrate this expression over all bodies producing craters of depth $h > z$, i.e., from $r' = z/\nu_1$ to $r' = r_1$.

For $\xi(h-z) \ll r_1$,

$$\int\limits_0^{\xi(h-z)} h_1 s_1 dr' = \frac{C'\nu_1}{4r^2}\int\limits_0^{\xi(h-z)} R^2\left(1 - \frac{R^2}{R_1^2}\right) r'^{1-p} dr' \approx \frac{C'\nu_1}{4r^2}\int\limits_0^{\xi(h-z)} R^2 r'^{1-p} dr' \approx \frac{C'\nu_1}{4r^2}\,\frac{[\xi(h-z)]^{4-p-2\alpha}}{(4-p-2\alpha)}\,\nu_{20}^2 r_0^{2\alpha} \tag{34}$$

and the ratio of integrals in the right-hand side of (33) is equal to $\xi(h-z) \times (4-p-2\alpha)/(5-p-2\alpha)$. Since for $r' \to r_1$ $\xi \approx 0.1 \ll 1$, the expression given for the ratio of the integrals is approximately suitable for the entire interval of r'. Consequently in view of (27) and (32) we obtain

$$K_1(z) \approx \frac{C'}{8r^2}\int\limits_{z/\nu_1}^{r_1} \left[\nu_1^2 r'^2 - z^2 - c_1(\nu_1 r' - z)^\gamma\right] \nu_2^2 r'^{2-p} dr', \tag{35}$$

where

$$c_1 = \frac{(4-p-2\alpha)\gamma\,\nu_1^{2-\gamma}}{5-p-2\alpha}\left[\frac{2\Delta R^2}{(p-1)\,\nu_{20}^2 r_0^{2\alpha}}\right]^{\gamma-1}, \quad \gamma = \frac{4-2\alpha}{3-2\alpha}. \tag{36}$$

Integrating we obtain the following expression for $z > \nu_1 r_0$:

$$K_1(\zeta) = C_1 \left\{ 1 - \frac{2}{p-3+2\alpha} \zeta^{5-p-2\alpha} + \frac{5-p-2\alpha}{p-3+2\alpha} \zeta^2 - c_2 [1 - (c_3 \zeta)^{3+\gamma-p-2\alpha}] \right\}, \qquad (37)$$

where

$$C_1 = \frac{C' \nu_1^2 \nu_{20}^2 r_0^{2m}}{8r^2 (5-p-2\alpha)} r_1^{5-p-2\alpha}, \quad c_2 = \frac{(5-p-2\alpha)\, c_1}{(3+\gamma-p-2\alpha)(\nu_1 r_1)^{2-\gamma}},$$

$$\zeta = \frac{z}{\nu_1 r_1}.$$

The coefficient c_3 inside the last bracket in (37) is slightly greater than unity. For $z < \nu_1 r_0$ the terms containing z in the expression for $K_1(z)$ will be different, but negligible compared with terms not containing z, which remain as before. At depths $z \ll r_1$ only the first term is significant. The factor c_2 in front of the square brackets is small compared with unity. Thus for $p = 3.5$, $\alpha = 0.15$, $r_0 = 1$ m, $\Delta R = 35$ km, $r_1 = 20$ km, $\nu_1 = 4$ and $\nu_{20} = 18$, the factor c_2 is 0.1. When one allows for the departure of the temperature march from linearity c_2 may increase slightly, but hardly more than by a factor of two, and therefore the role of the term associated with $T(h/2)$ is generally small.

To evaluate $K_2(z)$ we will adopt the following simplified scheme to describe mixing in impacts of the loosening type. As an increase in the size of falling bodies will not be accompanied by an increase in the rate of ejection of material, the rate of mixing l_m should be bounded. We assume that an elementary volume lying at a depth z before impact will lie with equal probability in the depth interval $(z - l_m,\ z + l_m)$ after impact. Furthermore, l_m should be of the order of the maximum height of lifting of material on impact. A reasonable value for l_m would be $l_m = \nu_1 r_1$, i.e., the maximum depth from which material was ejected during formation of the largest crater (before transition to a loosening explosion).

Reasoning in the same way as for $K_1(z)$, we find that the amount of heat (divided by $c\rho$) transported upward together with material across 1 cm^2 at a depth of z on impact of a body r' is given by

$$(h - z)\, T\left(\frac{h+z}{2}\right) \quad \text{for } h - z < l_m$$

and

$$l_m T\left(z + \frac{l_m}{2}\right) \quad \text{for } h - z > l_m.$$

h will be understood to mean the maximum depth from which material is lifted on impact. We will take, as before, $h = \nu_1 r'$. When the uplifted material sinks back, the amount of heat carried down across the same small area is

$$(h - z)\, T(h/2), \quad \text{if} \quad h - z < l_m, \quad z < l_m,$$
$$l_m T\left(\frac{z + l_m}{2}\right), \quad \text{if} \quad h - z > l_m, \quad z < l_m,$$
$$(h - z)\, T\left(\frac{h + z - l_m}{2}\right), \quad \text{if} \quad h - z < l_m, \quad z > l_m,$$
$$l_m T(z), \quad \text{if} \quad h - z > l_m, \quad z > l_m.$$

The resulting flux of heat to the exterior for one impact of scale h is given by

$$\frac{1}{2}\frac{\partial T}{\partial z}\min\{(h-z)\,l_m,\quad (h-z)\,z,\quad l_m l_m,\quad l_m z\},\tag{38}$$

where min $\{\ldots\}$ denotes the smallest of the four products in braces. As in the derivation of $K_1(z)$, we assume a linear march of temperature with depth as an approximation.

Summing this expression over impacts of all bodies larger than r_1 and dividing it by $\partial T/\partial z$, we obtain $K_2(z)$:

$$K_2(z)\approx\frac{1}{2}\int_{r_1}^{r_M}\min\{z\,(h-z),\quad zl_m,\ l_m(h-z),\ l_m l_m\}\,s\,dr'=$$

$$=\frac{C'}{8r^2}\int_{r_1}^{r_M}\min\{r\,(h-z),\ zl_m,\ l_m(h-z),\ l_m l_m\}\,v_2^2 r'^{2-p}dr'.\tag{39}$$

Let us take $l_m=v_1 r_1$, and $v_2=\text{const}=v_{20}\left(\frac{r_0}{r_1}\right)^n$ and set $z/v_1 r_1=\zeta$. Then for $\zeta<1$

$$K_2(z)=\frac{C'}{8r^2}v_2^2 z\left[\int_{r_1}^{r_1+z/v_1}(v_1 r'-z)\,r'^{2-p}dr'+l_m\int_{r_1+z/v_1}^{r_M}r'^{2-p}dr'\right];$$

$$K_2(\zeta)=C_2\zeta\left[\frac{1}{4-p}(1+\zeta)^{4-p}-\zeta-\frac{p-3}{4-p}-\left(\frac{r_1}{r_M}\right)^{p-3}\right],\tag{40}$$

where

$$C_2=\frac{C'v_1^2v_2^2r_1^{5-p}}{8\,(p-3)\,r^2}.\tag{41}$$

Similarly for $1<\zeta<r_M/2r_1$ we obtain

$$K_2(\zeta)=C_2\left\{\frac{1}{4-p}[(1+\zeta)^{4-p}-\zeta^{4-p}]-\left(\frac{r_1}{r_M}\right)^{p-3}\right\}\tag{42}$$

and for $r_M/2r_1<\zeta<r_M/r_1$

$$K_2(\zeta)=C_2\left[\frac{p-3}{4-p}\left(\frac{r_M}{r_1}\right)^{4-p}-\frac{1}{4-p}\zeta^{4-p}+\zeta\left(\frac{r_1}{r_M}\right)^{p-3}\right].\tag{43}$$

The complete coefficient of thermal conductivity $K(z)$ should also include the ordinary thermal conductivity $k(z)$ (molecular, radiant, etc.):

$$K(z)=K_1(z)+K_2(z)+k(z).$$

However the role of $k(z)$ becomes significant only at depths z close to $v_1 r_M$, where the sources of energy $\mathcal{E}$ are insignificant and $T(z)$ practically ceases to increase with z. Therefore $k(z)$ has little influence on temperature.

To make sure that the value obtained for $K(z)$ is sound, it would be desirable to evaluate it in some other, independent way. In principle the process of mixing by impacts from falling bodies is somewhat similar to the turbulent mixing of fluids. There heat transfer is usually described by means of the coefficient of "turbulent thermal conductivity" (Landau and Lifshits, 1953, p. 252):

$$K_{\text{turb}}\sim lv_{\text{turb}},$$

where l is the characteristic dimension (usually maximal). This gives K_{turb} only up to a constant factor (of the order of one) which is determined experimentally. In our case experimental determination is impossible. On the o other hand, thermal conduction is essentially a diffusion process. In gases, for example, the coefficients of thermal conductivity, diffusion and kinematic viscosity are identical and have the same form as K_{turb}:

$$k = D = \nu = \frac{1}{3} v\lambda = \frac{1}{3} \lambda^2/\tau, \tag{44}$$

where λ and τ are the mean free path and time of the molecules and v is their mean velocity. In cratering mixing, due to the smallness of the coefficient of ordinary molecular thermal conductivity, heat transfer takes place mainly together with transfer of material. The latter is random in character and can be described by the methods of the theory of random motions.

In the simplest case of random one-dimensional displacements of a particle along a straight line, by the same distance l every time (length of step), with equal probability in both directions and with frequency n, the mean square displacement of the particle from its initial position within the time t is given by

$$\sqrt{\overline{x^2}} = \sqrt{2Dt}, \tag{45}$$

where D is the coefficient of diffusion, which is given by

$$D = \frac{1}{2} nl^2 \tag{46}$$

(see Chandrasekhar, 1943). If a particle describes displacements of varying length l_i with frequency n_i, then

$$D = \frac{1}{2} \sum n_i l_i^2 = \frac{1}{2} n\overline{l^2}, \tag{47}$$

where $n = \sum n_i$.

From this, in particular, it is easy to obtain the expression of D written above for a gas by noting that

$$\overline{l^2} = \frac{1}{3} \overline{\lambda^2} = \frac{2}{3} \bar{\lambda}^2 = \frac{2}{3} \lambda^2 \quad \text{and} \quad n = 1/\tau.$$

Thus in the mixing of material the mean value of the coefficient of thermal conductivity can be taken in the form (47):

$$K' = \frac{1}{2} \sum n_i l_i^2 = \frac{1}{2} n\overline{l^2}. \tag{48}$$

It should be stressed that l should be interpreted in our case neither as the absolute displacement of a volume element of material along z, nor as the variation of its distance from the actual (not mean) surface, which changes position stepwise in each impact. Since we are interested in mixing in the sense of temperature equalization, l should be the measure of the temperature difference between volumes of material being mixed.

When a crater of depth $h=\nu_1 r'$ is formed, most of the material is ejected to the surface, i. e., for an element at depth z the scale $l\sim z$. A significant fraction of the material spills on the bottom from the edges of the crater. For the spilled material $l\sim h-z$. Some of the ejected matter lands at the bottom of deep craters that in some cases have not cooled down. To some extent this increases l for small z and decreases it for large z. On the average it seems one can assume

$$l^2 \approx \lambda h^2 \approx \overline{z^2} = \frac{1}{h}\int_0^h z^2 dz = \frac{1}{3}h^2. \tag{49}$$

Integrating over all craters of depth $z<h<\nu_1 r_1$, we obtain

$$K_1'(z) \approx \frac{1}{2}\sum n l^2 = \frac{\lambda}{2}\int_{z/\nu_1}^{r_1} h^2 s dr' = \frac{\lambda \nu_1^2 \nu_{20}^2 r_0^{2\alpha} C'}{8r^2}\int_{z/\nu_1}^{r_1} r'^{4-p-2\alpha} dr' = \lambda C_1\left[1-\left(\frac{z}{\nu_1 r_1}\right)^{5-p-2\alpha}\right]. \tag{50}$$

This expression differs from our earlier expression (37) for $K_1(z)$ only in a factor λ and in the form of the factor in brackets. As no account is taken in $K_1'(z)$ of ejected material reaching deeper craters and contributing to $\overline{l^2}$, the agreement between K_1 and K_1' can be regarded as satisfactory.

In impacts of large bodies with $r'>r_1$, it is presumed that a volume element can shift upward or downward by any distance less than l_m. Then $\overline{l^2}=l_m^2/3$ and for $z<\nu_1 r_1$

$$K_2'(z) = \frac{l_m^2}{3}\int_{r_1}^{r_M} s dr' = \frac{C'\nu_1^2 r_1^2 \nu_{20}^2 r_0^{2\alpha}}{4r^2\cdot 3r_1^{2\alpha}}\int_{r_1}^{r_M} r'^{2-p} dr' = \frac{2}{3}C_2\left[1-\left(\frac{r_1}{r_M}\right)^{p-3}\right], \tag{51}$$

while for $\nu_1 r_1 < z < \nu_1 r_M$

$$K_2'(z) = \frac{l_m^2}{3}\int_{z/\nu_1}^{r_M} s dr' = \frac{2}{3}C_2\left[\left(\frac{\nu_1 r_1}{z}\right)^{p-3}-\left(\frac{r_1}{r_M}\right)^{p-3}\right]. \tag{52}$$

Comparing these expressions with (40) — (43), we see that the differences between K_2 and K_2' are also relatively slight. Figure 12 shows the values of K_1 and K_2, K_1', K_2' and their sums $K=K_1+K_2$ and $K'=K_1'+K_2'$ as a function of $\zeta=z/\nu_1 r_1$ and corresponding to $p=3.5$, $r_1=40$ km, $r_M=100$ km, and $r_0=1$ m. It is interesting to note that the difference between K and K' is considerably less than that between individual components. The agreement is even excessive in view of how different the methods used to evaluate K and K' are, and how many simplifications they contain. It gives us ground to hope that the principal features of heat transfer in impact mixing have been elucidated correctly.

In neither method, however, was allowance made for crater overlap. As a result of overlap the mixing depth increases while the function $K(z)$ becomes, as it were, "blurred," increasing for small and large z. Sufficiently accurate allowance for this factor would require cumbersome calculations, on which we will not dwell here. We merely note that a very approximate estimate causes $K_1(0)$ to roughly double. The corresponding function $K(z)$ used to calculate terrestrial temperatures for the foregoing values of the parameters is represented in Figure 12 by a solid line.

Another major factor governing the warming of the growing Earth is the depth distribution of the energy liberated during impacts from falling bodies. It can be determined by the following methods:

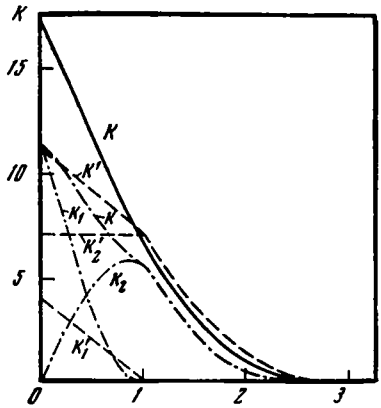

FIGURE 12. Coefficients of thermal conductivity as determined by different methods: K — from the expression for the heat flux; K' — by analogy with the coefficient of diffusion.

1. A fraction η_i of the total energy goes to the bottom of the crater (more precisely into the material not thrown out of the crater). Propagating with the shock wave, it is expended chiefly in warming and destroying material. A small part η_s of this energy goes to great depths in the form of an elastic seismic wave. This par amounts to about 1% of the entire impact energy (Bune, 1956; Pasechnik et al., 1960; O'Brien, 1960; Kirillov, 1962). We note that 1% of the Earth's gravitational energy corresponds to warming of the Earth by 400°. But estimates of η_s are not reliable enough.

2. A fraction of the total energy $\eta_e = 1 - \eta_i$ is thrown out of the crater together with the material. Most of it (of the order of η_i in quantity) converts into heat, a fraction η_m converting into energy of motion of ejected material and a fraction η_{ev} (apparently small) being expended in evapc ration of the material.

The main heat sources of the Earth are the large bodies. The role of the small bodies reduces chiefly to participation in the heat balance of the surface layer of the Earth, i.e., to the creation of a definite surface temperature T_{s0}. It is therefore most important to establish the depth distribution of the energy liberated in impacts of large bodies.

In the flat one-dimensional problem the simplest form of wave damping is exponential. It also gives an exponential function for the generation of the energy $E(z)$ per unit volume

$$J = J_0 e^{-bz}, \quad -\frac{dJ}{dz} = E(z) = bE_0 e^{-bz},$$

where $E_0 = \int_0^\infty E(z)\,dz$ is the energy liberated at all depths (in a column of 1 cm^2 cross section).

For a spherical wave (concentrated impact) in the simplest instance of damping (constant absorption coefficient) we have

$$J(r) = \frac{J_0}{r^2} e^{-br}, \quad \varepsilon(r) = b\frac{J_0}{r^2} e^{-br}. \tag{53}$$

To obtain the energy $E(z)$ liberated in a unit layer at depth z, one must integrate $\varepsilon(z)$ over all r for given constant z. Let

$$z = r\cos\theta, \quad \rho = r\sin\theta.$$

Then

$$r d\theta = \cos\theta d\rho$$

and

$$E(z)=\int_0^\infty \varepsilon(r)\,2\pi\rho d\rho=2\pi bJ_0\int_0^{\pi/2} e^{-\frac{bz}{\cos\theta}}\operatorname{tg}\theta d\theta=2\pi bJ_0\int_{bz}^\infty \frac{e^{-\chi}d\chi}{\chi}.$$

Since all the energy going into the hemisphere is given by $E_0=2\pi J_0$, we have

$$E(z)=bE_0\mathrm{E}_1(bz), \tag{54}$$

where

$$\mathrm{E}_1(x)=\int_x^\infty e^{-x}\frac{dx}{x}.$$

The inverse of b, which represents the characteristic wave damping distance, can be taken to be proportional to r': $b=1/\beta r'$. If we estimate b from the condition that 1% of the entire energy escapes below the zone of destruction ($\approx 2h$), we obtain $\beta\approx 2$.

The distribution of the energy liberated in the Earth in the fall of bodies of various sizes can then be written as

$$E(z)=\frac{v_0^2}{2}\left[\int_{r_m}^{r_1}\frac{\eta_i m'}{\beta_1 r'}\mathrm{E}_1\left(\frac{z}{\beta_1 r'}\right)n(r')\,dr'+\int_{r_m}^{r_1}\frac{(\eta_e-\eta_{ev})\,m'}{\beta_2 h_1}\times\right.$$
$$\left.\times\,\mathrm{E}_1\left(\frac{z}{\beta_2 h_1}\right)n(r')\,dr'+\int_{r_1}^{r_M}\frac{(1-\eta_{ev})\,m'}{\beta_3 r'}\mathrm{E}_1\left(\frac{r}{\beta_3 r'}\right)n(r')\,dr'\right], \tag{55}$$

where v_0 is the impact velocity, assumed to be the same for all bodies; h_1 is the thickness of the layer produced by material thrown out of the crater (see (14)), r_1 is the body radius at which "loosening" impacts set in (with nearly all the material falling back into the crater); $\beta_1\approx\beta_3\approx 2$, $\beta_2\approx 1$. The fraction of energy η_{ev} expended in evaporation does not exceed a few percents and will therefore be disregarded. The second integral is very cumbersome. For purposes of an approximate numerical estimate of $E(z)$, it is desirable to simplify it. Since for $r'\ll r_1$ the layer thickness h_1 is very small compared with r', the heat liberated inside this layer is almost entirely emitted from the surface, without contributing to the warming of the Earth's interior. Only for r' approaching r_1 does the second integral become comparable with the first. Let us therefore take

$$E(z)\approx\frac{v_0^2}{2}\left[\int_{r_m}^{r_1}\frac{(\eta_i+\eta_e r'/r_1)\,m'}{\beta_1 r'}\mathrm{E}_1\left(\frac{z}{\beta_1 r'}\right)n(r')\,dr'+\int_{r_1}^{r_M}\frac{m'}{\beta_3 r'}\mathrm{E}_1\left(\frac{z}{\beta_3 r'}\right)n(r')\,dr'\right]. \tag{56}$$

If $n(r')$ gives the number of bodies of radius r' falling per second over the entire Earth, $E(z)$ should also refer to the whole Earth. Given a power law of distribution of the falling bodies $n(r')=C'r'^{-p}$, the energy liberated per unit volume is given by

$$\varepsilon(z)=\frac{E(z)}{4\pi r^2}\approx\frac{v_0^2}{2}\frac{C'b'}{3r^2}\left[\int_{r_m}^{r_1}\frac{\eta_i+\eta_e r'/r_1}{\beta_1}\mathrm{E}_1\left(\frac{z}{\beta_1 r'}\right)r'^{2-p}dr'+\int_{r_1}^{r_M}\frac{1}{\beta_3}\mathrm{E}_1\left(\frac{z}{\beta_3 r'}\right)r'^{2-p}dr'\right], \tag{57}$$

where r is the Earth's radius. Quantity C' is determined from (28). For $p<4$ the quantity $\mathcal{E}$ in the equation of thermal conduction (51) is given by

$$\mathcal{E}(z)=\frac{\varepsilon(z)}{c\delta}=\frac{(4-p)\,v_0^2\dot{r}}{2cr_M^{4-p}}\left[\frac{1}{\beta_1}\int\limits_{r_m}^{r_1}(\eta_i+\eta_e r'/r_1)\,r'^{2-p}dr'\int\limits_{z/\beta_1 r'}^{\infty}\frac{e^{-\chi}d\chi}{\chi}+\frac{1}{\beta_3}\int\limits_{r_1}^{r_M}r'^{2-p}dr'\int\limits_{z/\beta_3 r'}^{\infty}\frac{e^{-\chi}d\chi}{\chi}\right], \quad (58)$$

where c is the heat capacity of the material of the Earth. We will take $\eta_1=\text{const}$, $\eta_e=\text{const}$, and $\beta_1=\beta_3=\beta$.

By changing the order of integration the double integrals can be reduced to single integrals. We set $z/\beta r'=\chi$, $z/\beta r_m=\chi_m$ and so forth. Then

$$\int\limits_{r_m}^{r_1}r'^{2-p}dr'\int\limits_{\chi}^{\infty}\frac{e^{-\chi}d\chi}{\chi}=\int\limits_{\chi_1}^{\chi_m}\frac{e^{-\chi}d\chi}{\chi}\int\limits_{z/\beta\chi}^{r_1}r'^{2-p}dr'+\int\limits_{\chi_m}^{\infty}\frac{e^{-\chi}d\chi}{\chi}\int\limits_{r_m}^{r_1}r'^{2-p}dr'=$$

$$=\frac{1}{p-3}\left[\left(\frac{\beta}{z}\right)^{p-3}\int\limits_{\chi_1}^{\chi_m}\frac{e^{-\chi}d\chi}{\chi^{4-p}}-r_1^{3-p}\mathrm{E}_1(\chi_1)+r_m^{3-p}\mathrm{E}_1(\chi_m)\right];$$

$$\int\limits_{r_m}^{r_1}r'^{3-p}dr'\int\limits_{\chi}^{\infty}\frac{e^{-\chi}d\chi}{\chi}=\frac{1}{4-p}\left[r_1^{4-p}\mathrm{E}_1(\chi_1)-r_m^{4-p}\mathrm{E}_1(\chi_m)-\left(\frac{z}{\beta}\right)^{4-p}\int\limits_{\chi_1}^{\chi_m}\frac{e^{-\chi}d\chi}{\chi^{5-p}}\right]$$

and

$$\mathcal{E}(z)\approx\frac{(4-p)\,v_0^2\dot{r}}{2\beta cr_M}\left\{\chi_M^{3-p}\left(\frac{1}{p-3}\left[\int\limits_{\chi_M}^{\chi_1}\frac{e^{-\chi}d\chi}{\chi^{4-p}}+\eta_i\int\limits_{\chi_1}^{\chi_m}\frac{e^{-\chi}d\chi}{\chi^{4-p}}\right]-\right.\right.$$

$$\left.-\frac{\eta_e\chi_1}{4-p}\int\limits_{\chi_1}^{\chi_m}\frac{e^{-\chi}d\chi}{\chi^{5-p}}\right)+\left(\frac{1-\eta_i}{p-3}+\frac{\eta_e}{4-p}\right)\left(\frac{r_M}{r_1}\right)^{p-3}\mathrm{E}_1(\chi_1)-$$

$$\left.-\frac{1}{p-3}\mathrm{E}_1(\chi_M)+\left(\frac{\eta_i}{p-3}-\frac{\eta_e}{4-p}\frac{r_m}{r_1}\right)\left(\frac{r_M}{r_m}\right)^{p-3}\mathrm{E}_1(\chi_m)\right\}. \quad (59)$$

In particular, for $p=3.5$

$$\mathcal{E}(z)=\frac{v_0^2\dot{r}}{2\beta cr_M}\left\{\sqrt{\frac{\pi}{\chi_M}}\left[(1-\eta_i-2\eta_e\chi_1)\operatorname{erf}\sqrt{\chi_1}+(\eta_i+2\eta_e\chi_1)\operatorname{erf}\sqrt{\chi_m}-\operatorname{erf}\sqrt{\chi_M}\right]-\right.$$

$$-\frac{2\eta_e\chi_1}{\sqrt{\pi}}\left(\frac{e^{-\chi_1}}{\sqrt{\chi_1}}-\frac{e^{-\chi_m}}{\sqrt{\chi_m}}\right)+(1+\eta_e-\eta_i)\sqrt{\frac{r_M}{r_1}}\mathrm{E}_1(\chi_1)+$$

$$\left.\left(\eta_i-\eta_e\frac{r_m}{r_1}\right)\sqrt{\frac{r_M}{r_m}}\mathrm{E}_1(\chi_m)-\mathrm{E}_1(\chi_M)\right\}. \quad (60)$$

Since $p<4$, one can take $r_m=0$ and $\chi=\infty$. Then for $p=3.5$

$$\mathcal{E}(z)=\frac{v_0^2\dot{r}}{2\beta cr_M}\left\{\sqrt{\frac{\pi}{\chi_M}}\left[(1-\eta_i-2\eta_e\chi_1)\operatorname{erf}\sqrt{\chi_1}+\eta_i+\right.\right.$$

$$\left.\left.+2\eta_e\chi_1-\operatorname{erf}\sqrt{\chi_M}\right]-2\eta_e\sqrt{\frac{r_M}{r_1}}e^{-\chi_1}+(1-\eta_i+\eta_e)\sqrt{\frac{r_M}{r_1}}\mathrm{E}_1(\chi_1)-\mathrm{E}_1(\chi_M)\right\}. \quad (61)$$

The functions $K(z)$ and $\mathcal{E}(z)$ which we obtained permit us to calculate Earth warming due to impacts from large bodies (first term in (12)). Unfortunately there are not enough data to allow us to dwell on definite values of the

parameters in the formulas for $K(z)$ and $\mathcal{E}(z)$. A calculation carried out for $p = 3.5$, $r_1 = 40$ km, $r_M = 100$ km, $\eta_i = 0.4$ and $\eta_e = 0.6$ indicates that warming due to large bodies amounts to about 1100° at a depth of 400 — 500 km below the Earth's surface. Warming increases with increasing r_M and decreases with increasing r_1 and p.

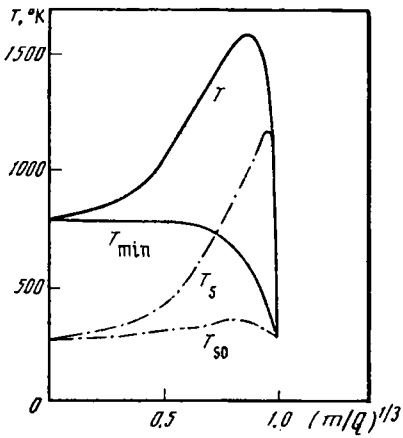

FIGURE 13. Primary temperature of the Earth:

T_{s0} — warming of Earth during growth process by impacts from small bodies and particles; T_{min} — corresponding initial Earth temperature with allowance for contraction and radioactive heating during growth process (100 million years); T_s — warming of Earth by impacts of bodies of various size, including large ones; T — corresponding initial Earth temperature.

The maximum warming at depths of 400 — 500 km due to impacts of large bodies will be denoted by T_M. From (61) it is apparent that the energy liberated in impacts is proportional to $v_0^2 \propto m^{2/3}$. For smaller m, moreover, sizes of falling bodies were also smaller. The liberated energy penetrated to lesser depths and less of it remained inside the Earth. Body sizes r_M can be assumed to be proportional to $r \propto m^{1/3}$, while their depth of penetration $w \propto r_M v_0^{2/3} \propto m^{4/9}$ (Andriankin and Stepanov, 1963). One can assume approximately that the warming of the Earth due to body impacts was proportional to m^u, where u apparently lies between 1 and 2. Then from (13.12) the distribution of the primary temperature inside the Earth can be written as

$$T(m) = \left[T_{s0} + T_M\left(\frac{m}{Q}\right)^u\right] A(m) + B(m)\,\varepsilon(t - t_s), \tag{62}$$

where T_{s0} is the warming of the Earth due to impacts of small bodies and particles, determined according to (62).

We noted earlier that calculations point to values of about one thousand degrees or possibly slightly more. Assuming $T_M = 1000°$ and $u = 1$, we obtain the distribution of the primary terrestrial temperature T given in Figure 13. For comparison we give the curve of the temperature T_{s0}, obtained under the assumption that all the material landing on Earth during its formation consisted of small bodies and particles, and the corresponding curve of the initial temperature T_{min}. We see that the maximum initial temperature occurred in the region of the upper mantle and may have exceeded 1500°K. For $T_M = 1200°$ it is nearly 2000°K. The importance of this preliminary result for the study of the Earth's thermal history makes it imperative that the parameters appearing in the foregoing formulas and governing the initial temperature of the Earth be made more accurate.

Chapter 16

PRIMARY INHOMOGENEITIES OF THE EARTH'S MANTLE

41. Inhomogeneities due to differences in chemical composition between large bodies

A number of recent, independent data indicate that there are pronounced horizontal inhomogeneities varying in scale and extending to various depths inside the Earth's mantle. Gravimetric maps clearly show positive and negative gravity anomalies covering areas several thousands of kilometers in diameter (Lyustikh, 1954). Analysis of zonal harmonics in the Earth's gravitational potential detected by satellite measurements (O'Keefe et al., 1959; King-Hele, 1962) has revealed (Munk and MacDonald, 1960; MacDonald, 1962) that the observed anomalies are considerably greater than the anomalies calculated for the continents under the assumption of hydrostatic equilibrium (isostasy), and are opposite in sign. They could not be due to density variations in the crust and are undoubtedly caused by large-scale horizontal inhomogeneities in the Earth's mantle. Studies of tidal deformations of the crust (Parijsky, 1963) show that the elastic properties of the mantle in the European sector of the USSR and in Central Asia are different. Electromagnetic and seismic observations also point to the existence of regional inhomogeneities in the Earth's mantle (Tikhonov et al., 1964; Fedotov and Kuzin, 1963).

The presence of the oceans and continents is also evidence of large horizontal inhomogeneities. It is hardly likely that such large formations could have developed in a primordially quasihomogeneous Earth. Their existence is rather to be viewed as evidence that the Earth's mantle contained large-scale primary inhomogeneities.

Evidence indicating the probable existence of large-scale primary inhomogeneities inside the mantle is also provided by the study of the Earth's formation. We noted in Chapter 8 that large bodies must have constituted a considerable fraction of the mass of solid material from which the Earth was formed. In Chapters 9 and 11 it was shown that the masses of the largest bodies falling on the Earth were of the order of one thousandth of the Earth's mass. A striking illustration of the important role of large bodies in the formation of the planets and their satellites is provided by the lunar craters and seas. The lunar seas were formed as a result of "planetesimals" a few tens of kilometers in diameter striking the Moon. Additional masses (mascons) inside them detected recently from gravity anomalies point to considerable inhomogeneity of the Moon's outer layers. Many large craters have also been discovered on Mars (Leighton et al., 1965).

There are two possible types of inhomogeneity traceable to large bodies striking the Earth: inhomogeneities arising from differences in the chemical composition of incoming bodies, and inhomogeneities due to impacts accompanied by liberation of large amounts of energy (Safronov, 1964 b; 1965b). Let us begin with the former.

The chemical composition of the planets varies regularly with distance from the Sun. The inner planets are denser than the outer ones. Urey, Elsasser and Rochester (1959) record density differences of up to 0.2 g/cm^3 in stony meteorites, due probably to the fact that their parent bodies originated in different regions of the asteroid zone. Variations have also been detected in the composition of iron meteorites. At standard pressure Mercury has the densest material in the solar system. The density of Venus is apparently several percents higher than that of the Earth (Kozlovskaya, 1966). The source zone of the newly developed Earth extended nearly from Venus' orbit to that of Mars. One might expect that bodies formed in different parts of this broad zone would display variations of several percents in composition and density.

If the bodies were incorporated in the Earth roughly in their original form as local inclusions, they must have introduced pronounced inhomogeneities in the mantle and given rise to motions inside it. Given a resistance threshold of 10 to 10^2 bar for the mantle material, inclusions several tens of kilometers in diameter having a density that differed from that of the surrounding material by 0.1 g/cm^3 must have sunk or floated under the influence of gravitational forces. Larger inclusions could have shifted starting from smaller density differentials. For sufficiently large inclusions, differentiation could have begun directly after formation of the Earth. Smaller inclusions would have begun to shift only after some degree of warming had taken place and the viscosity of the surrounding mantle material had decreased.

Differences in the composition of the bodies may have been reflected in differences in the content of radioactive elements (again amounting to a few percents), as well as in density variations. The two may have coexisted, but unlike the latter, the former did not manifest themselves immediately. Inclusions containing an excess of radioactive elements warmed up somewhat more rapidly in the process of decay, gradually becoming less dense than the surrounding material. In these regions partial melting of silicates, their uplift and the formation of the crust must have begun earlier. Magnitskii (1960) has attempted to explain existing gravity anomalies with the help of inclusions several hundreds of kilometers in diameter and having an excess of radioactive elements.

As has already been mentioned, these inhomogeneities associated with variations in the bodies' chemical composition would have occurred if the bodies became incorporated in the Earth in the form of local inclusions and did not disperse on striking the Earth. The velocities of the falling bodies were only slightly in excess of the parabolic velocity ($v = v_e\sqrt{1 + 1/2\theta}$), but at the closing stage of the Earth's growth they nevertheless amounted to 10 — 12 km/sec. Impacts of such velocity would lead to disintegration of the bodies and scattering of their material over a large area together with the material thrown out of the crater. This means that inhomogeneities associated with differences in composition must have been largely smoothed over.

In experimental studies of a tinted droplet in a container of water, the falling droplet spreads out over the surface of the "crater". In certain cases the "crater" collapses and a cumulative jet ending in a droplet containing almost all the water of the original tinted droplet rises upward from its center (Charters, 1960). Attempts have been made to attribute the presence of central mounds inside many lunar craters to a similar sliding of material from the edges of the crater toward the center (Shoemaker, 1962). If the central mounds really consisted to a large extent of fragments of a fallen body, the inhomogeneities introduced by the bodies should be fairly evident. But the brittleness of rocks makes them very different from liquids and metals. Charters notes that no fragments of the projectile were found at the center of the crater in experiments with rocks. It seems that bodies of silicate composition scatter on falling at high velocities.

However, the situation changes drastically in the case of impacts of large bodies. It was shown in Chapter 14 that when very large bodies fall gravitation becomes important and geometric similitude is destroyed. Qualitatively the picture is as follows. As the size of the falling bodies increases, the depth of penetration ("explosion" depth) increases accordingly. But since the energy per unit mass does not change with increasing size (same rate of fall), the initial scatter velocity of the material and correspondingly the scatter distance remain as before in the first approximation. Therefore for sufficiently large body sizes the radius of the crater formed in the case of geometric similitude, would exceed the distance of scatter of material from the crater. As a result nearly all the ejected material would fall back into the crater and all the energy liberated on impact would be trapped together with the material inside the crater. The fall of such large bodies is similar to camouflets or loosening explosions. The material of the falling body is not scattered in this case, but remains within a closed volume exceeding the volume of the body itself by no more than one order. A deviation of a few percents on the part of the chemical composition of the body from that of the Earth will cause a volume equal to several body volumes to deviate in composition by tenths of a percent and correspondingly to deviate in density from the mean density of Earth material by ~ 0.01 g/cm^3. Such an inclusion will cross the resistance threshold and sink (or float up) if the body producing it measures about 100 km in diameter or more.

42. Inhomogeneities due to impacts of falling bodies

As a result of the disintegration of the material adjacent to the crater by the shock wave, and also due to ejected material falling back, a layer of crushed rock (breccia) is formed under the crater. Data obtained by drilling in the Holliford and Brent craters agree with Rottenberg's theoretical conjecture that in granite gneisses the depth of the breccia in the central portion of a crater should amount to about one third of the crater diameter (Beals et al., 1960; Innes, 1961). In the disintegration zone the pressure along the front of the shock wave was greater than the rocks' resistance and oscillations propagated inelastically. In the first approximation the volume of crushed rock is proportional to the total impact energy of the body (Innes, 1961; Baldwin, 1963). The density of material in the breccia region is lower than in adjacent regions where the material was not subjected to

disintegration. According to drilling data, confirmed by independent estimates based on the measured values of gravity anomalies, the density difference averages 0.2 g/cm^3 (Innes, 1961). The volume of the breccia is several times greater than that of the crater and therefore much greater than that of the fallen body. Density inhomogeneities stemming from the disintegration of material in impacts are greater than those discussed in Section 41. The foregoing data, however, refer to depths of the order of 1 km. At depths of interest to us (of the order of hundreds of kilometers) the density drop after impact should obviously be substantially less, and its neutralization should proceed more rapidly.

More definite statements can be made regarding thermal inhomogeneities produced in the fall of large bodies. A considerable fraction of the shock wave energy converts into heat in the breccia zone. Öpik (1958) worked out an approximate scheme for the impact mechanism which he used to evaluate warming up of material inside the impact zone. For a rate of fall of 10 km/sec, this warming amounts to 580°, 208° and 93° along the frontal surface of a wave occupying, respectively, a 30-fold, 50-fold and 75-fold mass of the falling body. Although this estimate is apparently somewhat exaggerated, it shows that a mass of material considerably greater than the mass of the body itself will warm up by hundreds of degrees. The thickness of the warmed layer is of the order of the diameter of the fallen body. Experimental data show broad differences between different rocks as regards warming on impact (Chao, 1967). Thus quartz requires a peak pressure of 400 kbar to warm up by 600°, while sandstone needs only 90 kbar. The pressures necessary for warming by 1500° are respectively about 500 and 200 kbar.

Only the largest of the bodies could have caused significant temperature inhomogeneities. First, for any reasonable distribution function the number of bodies decreases with increasing size. Therefore the deviation of any random quantity from its mean value of $1:\sqrt{N}$ will increase. Second, it is only when very large bodies are involved that nearly all the energy liberated on impact will remain in the impact region. Third, only the largest bodies were capable of warming a layer so thick that the lower portion of the layer was situated below the mixing region due to impacts of other bodies and was not subjected to the effective cooling associated with this mixing. Fourth, the larger the fallen body, the larger the region heated and the greater the time required for the temperature of this region to level down to that of the surrounding medium.

An idea of the body sizes capable of giving rise to long-lived thermal inhomogeneities can be formed by estimating the rate of cooling of a plane layer near the Earth's surface (Safronov, 1965b). For the initial and boundary conditions

$$T(x, 0) = \begin{cases} T_1 & 0 < x < h \\ 0 & x > h \end{cases} \qquad T(0, t) = 0 \tag{1}$$

the solution of the equation of thermal conduction without sources for a plane half-space has the form

$$T(x, t) = \frac{T_1}{2}\left[2\,\mathrm{erf}(y) - \mathrm{erf}(y - y') - \mathrm{erf}(y + y')\right], \tag{2}$$

where $y = x/2\sqrt{kt}$ and $y' = h/2\sqrt{kt}$. The temperature at the middle of the layer $x = h/2$ is given by

$$T_e = T\left(\frac{h}{2}, t\right) = \frac{T_1}{2}\left[3\,\mathrm{erf}\left(\frac{y'}{2}\right) - \mathrm{erf}\left(3\frac{y'}{2}\right)\right]. \qquad (3)$$

Below we give the ratio T_e/T_1 for various layer thicknesses h, 10^9 years after cooling begins, for $k = 0.01$:

h, km	50	100	200	300	500
T_e/T_1.	0.002	0.012	0.080	0.21	0.52

At the center of a layer 300 km thick, one billion years later about 20% of the original temperature excess is still left (i. e., about 100° if the layer was warmed by 500°). Consequently bodies several hundreds of kilometers in diameter must have induced considerable thermal inhomogeneities in the developing Earth while falling. Larger inhomogeneities lasted 1 — 2 billion years, i. e., until such time as intensive warming by radioactive heat had caused the viscosity of the Earth's material to alter appreciably, with gravitational differentiation setting in. In sections of the upper mantle with temperature excesses of 100 — 150° stemming from impacts of large bodies, the crust material must have begun to melt 100 million years earlier than in other zones.

Recently it has emerged that the energy of lunar tides inside the solid Earth may have dissipated preferentially in warmer sections of the upper mantle with material of low elasticity, imparting to these sections a distinct additional warming (Ruskol, 1965). Thanks to this energy source thermal inhomogeneities could have survived over longer periods, possibly even increasing in intensity. The impact areas of the largest bodies striking the Earth during its formation could have converted into relatively stable regions of higher temperature in which all processes associated with melting of the crust and tectonic activity began earlier and proceeded with greater intensity. These regions could have remained active for a long time, until such time as they had lost a considerable fraction of their radioactive elements by migration of the latter to the surface together with the light melts which produced the crust.

The most important inhomogeneities must have been due to bodies several hundreds of kilometers in diameter; their dimensions must have been in excess of one thousand kilometers. This fact permits us to conjecture that there may have been a connection between initial thermal inhomogeneities in the mantle (due to impacts of the largest bodies falling on the Earth) and the subsequent differences in the thermal history of these regions which led to the formation of the continents. Though many factors influenced the complex process of the Earth's evolution, it would seem that initial thermal inhomogeneities played a particularly important role and must be taken into account in any study of this process.

CONCLUSIONS

The theory of planet formation by accumulation of solid bodies and particles has provided the Earth sciences with important information about the initial state of the Earth and especially about its primary temperature. If all gravitational energy liberated in the Earth's formation had remained in its interior, the Earth would have warmed up to 40,000°. Therefore estimates of the primary temperature will depend essentially on how the Earth is assumed to have been formed. The old view that the planets were formed by the condensation of gaseous clusters, and that the Earth was originally in a hot, molten state, has long held sway. Schmidt's conversion to the idea of planet formation by fusion of solid bodies and particles led him to draw the important conclusion that the Earth was initially in a relatively cool state, a conclusion that had an important influence on the subsequent development of the Earth sciences. Given this method of formation, the main sources of heat during the period of the Earth's growth must have been impacts of falling bodies, the contraction of its material under the pressure of layers accumulating at the top, and the generation of radioactive heat. The warming resulting from contraction is proportional to the temperature of the material being compressed. In the mantle at the boundary of the core, contraction caused the initial temperature to increase 1.9 times; at the Earth's center (for a metallized silicate core) it caused an increase of 2.1 times, assuming that phase transition takes place without generation of heat. The total energy of contraction of the Earth amounts to $4.2 \cdot 10^{38}$ erg, of which $2.3 \cdot 10^{38}$ erg is expended in the phase transition. The conversion into heat of just one tenth of the transition energy would warm the core by 1000°. It would therefore be important to have an estimate of the amount of energy converting into heat in the phase transition.

Owing to the comparatively short time within which the Earth was formed (10^8 years), warming due to generation of radioactive heat over this period was limited: the inner part of the mantle was warmed 100°, the core 150 — 200°.

The main heat source of the growing Earth was the impacts of the bodies and particles from which it was formed. Estimates of the initial temperature depend largely on the body sizes assumed. The first calculations, based on the assumption that these bodies were small, produced a low initial temperature — from 300°K near the surface to 800 — 900°K at the center. The impact energy of small bodies and particles was liberated near the surface of the growing Earth, practically all of it being emitted into space. Even in the period of most intensive growth, the temperature of its surface would not have exceeded 350 — 400°K.

The importance of large bodies in the process of planet formation has recently begun to emerge (see Part II). The largest bodies falling on the Earth had diameters reaching several hundreds of kilometers. The larger the incident body, the greater the depth at which its impact energy was released and hence the greater the fraction of this energy trapped inside the Earth, unable to escape into space. On the other hand, larger bodies produced deeper craters, inducing more intensive mixing of the material on impact. Heat transfer by mixing of material during the impact of large bodies is far more efficient than heat transfer by ordinary thermal conduction. To evaluate the warming up of the growing Earth (i.e., an Earth with a mobile boundary) due to impacts of falling bodies using the equation of thermal conduction, it is necessary to determine the analog of the coefficient of thermal conductivity K and the depth distribution of the energy $\mathcal{E}$ liberated on impact. But evaluation of K and $\mathcal{E}$ requires further development of the theory of cratering, especially as regards the consequences of the fall of very large bodies (which contributed most to the Earth's primary temperature). Here the Earth's gravity essentially destroys geometric similitude, leading to qualitatively new phenomena.

The velocity of the bodies at the instant of impact did not depend on their size and at the concluding phase of growth amounted to 10 — 12 km/sec. The depth of penetration ("explosion" depth) is proportional in this case to the size of the incident body (about twice its diameter). The rate of ejection and therefore the distance of scattering of the ejected material do not depend on the body size. In the case of very large falling bodies, the scattering distance is less than the radius of the crater which would have formed for geometric similitude. Therefore the greater part of the ejected material falls back into the crater. Such impacts are similar to "loosening explosions." As the size of the falling bodies increases, the character of the crater alters in the same way as when the relative depth of explosion increases. Although the mass of material over the explosion center per unit mass of incident body remains constant, the energy required to eject a unit mass from the crater inside the Earth's gravitational field increases with the size of the crater. The relative explosion depth should therefore be measured not by the ratio $w/C^{1/3}$ but by the ratio $w/C^{1/\mu}$, where μ is close to 4 (for large bodies). The impacts of bodies with diameters exceeding one hundred kilometers are similar to loosening explosions. These are the impacts that warmed the Earth most efficiently, as nearly all the heat released when they fall remains buried inside the filled-in crater.

Numerical estimates show that the maximum of initial temperature of the Earth occurred in the region of the upper mantle and probably exceeded 1500°K. This means that the time required for the subsequent warming of the hotter regions to the melting point of low-melting substances and for initiation of the process of crust formation may have been short (less than one billion years). For a more exact estimate of the original temperature additional research is necessary on the theory of cratering (especially for large-body impacts) and on the size distribution function of the bodies from which the Earth was formed. Also required is the construction of a more rigorous theory of heat transfer in terrestrial material mixed by the impacts of falling bodies.

The largest of the bodies striking the Earth induces primary inhomogeneities in its mantle. One type of inhomogeneity was related to differences in chemical composition between large bodies. The bodies were formed

inside a broad zone between the orbits of Venus and Mars. The density and content of major chemical elements of these planets differ from those of the Earth by a few percents. The same order of differences in composition can be expected to prevail between individual bodies landing on the Earth. Small bodies scattered over a large area on falling. Very large bodies intermingle with a volume of material exceeding their own volume by only one order. Such inclusions could have deviated from the mean density by ~ 0.01 g/cm^3 for an initial deviation in body composition of several percents. For body diameters of over 100 km, such inclusions would overcome the resistance threshold of the terrestrial rocks and begin to sink or float (depending on the sign of the density deviation).

Another type of primary inhomogeneity, temperature inhomogeneities, was due to impacts of falling bodies. Unlike those just discussed, they could have been produced by large bodies of all compositions. Only the largest gave rise to significant inhomogeneities. The layer they warmed was so thick that its lower portion lay outside the zone of mixing by impacts of other bodies. Equalization of temperature proceeded slowly, and the inhomogeneities may have lasted 1 — 2 billion years. Preferential dissipation of the energy of lunar tides in these warmer regions could have converted them into relatively stable regions of higher temperature in which processes associated with crystal melting set in earlier and proceeded more intensively. These regions had dimensions in excess of one thousand kilometers, and it is natural to conjecture that the formation of the continents was related to their presence.

Thus while the theory of planet formation by accumulation of solid bodies and particles can furnish information of importance for the Earth sciences regarding the Earth's initial state, to obtain reliable results in this limited field calls for cooperation of scientists in a variety of specialities. The early history of the Earth still contains questions relating to the primary atmosphere, primary hydrosphere, and so on. The study of the initial state and evolution of the Earth is one of the most pressing problems of geophysics. Recently initiated comparative studies of the structure, composition and thermal history of the Earth and other planets may prove of great help in its solution.

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